

TRANSLATION SEMIGROUPS AND THEIR LINEARIZATIONS ON SPACES OF INTEGRABLE FUNCTIONS

ANNETTE GRABOSCH

ABSTRACT. Of concern is the unbounded operator $A_\Phi f = f'$ with nonlinear domain $D(A_\Phi) = \{f \in W^{1,1}: f(0) = \Phi(f)\}$ which is considered on the Banach space E of Bochner integrable functions on an interval with values in a Banach space F . Under the assumption that Φ is a Lipschitz continuous operator from E to F , it is shown that A_Φ generates a strongly continuous translation semigroup $(T_\Phi(t))_{t \geq 0}$. For linear operators Φ several properties such as essential-compactness, positivity, and irreducibility of the semigroup $(T_\Phi(t))_{t \geq 0}$ depending on the operator Φ are studied. It is shown that if F is a Banach lattice with order continuous norm, then $(T_{|\Phi|}(t))_{t \geq 0}$ is the modulus semigroup of $(T_\Phi(t))_{t \geq 0}$. Finally spectral properties of A_Φ are studied and the spectral bound $s(A_\Phi)$ is determined. This leads to a result on the global asymptotic behavior in the case where Φ is linear and to a local stability result in the case where Φ is Fréchet differentiable.

0. INTRODUCTION

One of the basic examples in the theory of one-parameter semigroups on function spaces are semigroups which are generated by the first derivative on a suitable domain.

At the beginning we want to review a particular simple example from the theory of strongly continuous semigroups which eventually will lead us to introduce a general notion of translation semigroups. We consider the unbounded operator A_0 given by $A_0 f = f'$ with domain $D(A_0) = \{f \in AC[-1, 0]: f(0) = 0\}$ in the Banach space of Lebesgue-integrable functions $E = L^1[-1, 0]$.

It is a straightforward computation to verify that A_0 is the generator of a strongly continuous semigroup $(T_0(t))_{t \geq 0}$ of linear operators explicitly given by the formula

$$(1) \quad [T_0(t)f](x) = \begin{cases} f(x+t) & \text{if } x+t \leq 0, \\ 0 & \text{if } x+t > 0. \end{cases}$$

Received by the editors July 9, 1987 and, in revised form, October 20, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 47D05, 47H20; Secondary 39B70, 45D05, 92A15.

Key words and phrases. Linear and nonlinear strongly continuous semigroups, translation semigroups, first derivatives, compactness, positivity, irreducibility, domination, spectral properties, asymptotics, linearization, stability of equilibrium solutions, functional equations, Volterra equations.

Informally one can say that the semigroup $(T_0(t))_{t \geq 0}$ shifts functions defined on the interval $[-1, 0]$ to the left, "pushing in" the value 0 at the right endpoint of the interval $[-1, 0]$.

With this idea in mind one verifies directly that $(T_0(t))_{t \geq 0}$ is a nilpotent semigroup, i.e., $T_0(t) = 0$ for $t > 1$ (which implies of course the compactness of $(T_0(t))_{t \geq 0}$). Another important property of $(T_0(t))_{t \geq 0}$ is the positivity, which is given since each operator $T_0(t)$ maps positive functions of $L^1[-1, 0]$ onto positive functions.

This simple example gives an idea of how first derivatives are connected to translations. It furthermore suggests the study of translations, where one "pushes in" values other than 0 from the right. These values may depend (linearly or nonlinearly) on the starting function.

The purpose of this paper is to state the correspondence of such translations with first derivatives and to study diverse properties such as compactness, positivity, and irreducibility of the semigroup. These kinds of considerations are crucial in determining stability properties and asymptotic behavior of these translation semigroups, or, in other words of solutions of the corresponding Cauchy problems. Linearization methods lead to stability properties of equilibria in the nonlinear case.

Finally we will give some examples in which translations play an essential role in describing the dynamics of natural systems and where our results can be applied. Of special interest are applications to functional equations, for example, renewal equations and Volterra equations.

Further applications and related considerations can be found in [18]. For the general correspondence between translation semigroups and first derivatives we refer also to [33 and 30]. Special translation semigroups on spaces of continuous functions were for the first time intensely studied by J. K. Hale [23] and afterwards by several others (see for example [14, 39, 7, 11, 8] and the references therein). The special aspects of positivity are discussed in [17, 25, 26 and 27] (see also [32, Section B-IV.3]). On spaces of integrable functions we refer to the references [10, 1, 3, 38, 40 and 41] which mention some of the many related considerations.

1. FIRST DERIVATIVES AND TRANSLATION SEMIGROUPS

In the introduction we observed that in a simple case the first derivative is the generator of a one-parameter semigroup of translations. This fact gives rise to the conjecture that the first derivative considered with a suitable domain on a suitable state space is always a generator of a semigroup having this property.

Indeed the following results by A. T. Plant [33], resp. F. Martello [30], confirm this idea for first derivatives on the space of continuous functions on a closed interval with values in a Banach space, resp. on the space of Bochner integrable functions with values in a Banach space. For first derivatives on the space of

continuous functions $C([-r, 0], F)$, where F is any Banach space and $r < \infty$, A. T. Plant showed the following result.

Proposition 1.1. *Let A be an operator on $C([-r, 0], F)$ ($r < \infty$) with $D(A) \subseteq C^1([-r, 0], F)$ and $Af = f'$ with the additional property that there exists w such that $A - w$ is m -dissipative (i.e. $A - w$ is dissipative and $\text{im}(\text{Id} - \alpha A) = E$ for small $\alpha > 0$). Then A is a generator of a strongly continuous semigroup which satisfies*

$$[T(t)f](s) = \begin{cases} f(t+s) & \text{if } t+s < 0, \\ [T(t+s)f](0) & \text{if } t+s \geq 0. \end{cases}$$

Led by this characterization F. Martello proved the corresponding result for semigroups on $L^1([-r, 0], F)$ where again F is a Banach space and $r < \infty$.

Proposition 1.2. *Let A be an operator on $L^1([-r, 0], F)$ ($r < \infty$) with $D(A) \subseteq W^{1,1} := \{f \in L^1([-r, 0], F) : f \text{ is absolutely continuous, almost everywhere differentiable and } f' \in L^1\}$ and $Af = f'$ with the additional property that there exists w such that $A - w$ is m -dissipative. Then A is a generator of a strongly continuous semigroup which satisfies $[T(t)f](s) = f(t+s)$ for s with $t+s < 0$.*

Martello proved this result by applying Proposition 1.1 to the restriction of A to the subspace $C([-r, 0], F)$ of $L^1([-r, 0], F)$.

For the rest of this paper we want to concentrate on the case where $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ ($\eta \geq 0$). The weight function $e^{\eta s}$ is introduced to include equilibrium solutions other than 0 in the discussion of local stability in §4.

Also it should be remarked that all of the following results remain valid if we consider our translation semigroups on $L^1([-r, 0], F)$ where $r < \infty$ instead. The formulations and proofs need only minor modifications. See [32 and 18] for a detailed discussion.

First we want to ask, conversely to the statement of Propositions 1.1 and 1.2, whether translation semigroups on $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ are generally generated by first derivatives. Although in §3 we will restrict our attention most often to linear translation semigroups, here we consider (as we have in the preceding two propositions) nonlinear strongly continuous semigroups and refer to the review article [15] and to the forthcoming book [16] of J. A. Goldstein for the corresponding definitions and results from the theory of nonlinear semigroups.

Definition 1.3. *A strongly continuous semigroup $(T(t))_{t \geq 0}$ on the Banach space $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ is called a translation semigroup if $[T(t)f](x) = f(x+t)$ for $x+t < 0$.*

(Here and in the following discussion, equality usually means a.e. equality.)

Our proof will follow closely the arguments of G. F. Webb [41] in the proof of his Proposition 3.1, where the following result is shown for a special type of translation semigroup.

Proposition 1.4. *Let $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ and let $(T(t))_{t \geq 0}$ be a translation semigroup on E . Let A be the infinitesimal generator of $(T(t))_{t \geq 0}$. Then $D(A) \subseteq W^{1,1}$ and for $f \in D(A)$ we have $Af = f'$.*

Furthermore the map $f \mapsto f(0)$ is continuous from $(D(A), \|\cdot\|_A)$ into F .

Proof. Let $f \in D(A)$ with $Af = \psi$. For almost every a, b with $a < b < 0$ we have

$$\begin{aligned} f(b) - f(a) &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_b^{b+t} f(x) dx - \int_a^{a+t} f(x) dx \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_{t+a}^{t+b} f(x) dx - \int_a^b f(x) dx \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_a^b [f(t+x) - f(x)] dx \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_a^b [T(t)f(x) - f(x)] dx \right) \\ &= \int_a^b \psi(x) dx. \end{aligned}$$

Hence

$$f(b) = f(a) + \int_a^b \psi(x) dx \quad \text{for almost every } a \in (-\infty, 0) \text{ and } b \in (a, 0).$$

Thus we can choose an absolutely continuous representative f , such that f is a.e. differentiable and $f' = \psi$, i.e., $f \in W^{1,1}$. This proves that $Af = \psi = f'$.

If $f \in W^{1,1}$ then we especially have that f is continuous and thus $f(0)$ exists. Furthermore the following estimate holds:

$$\begin{aligned} \|f(0)\|_F &= \left\| \frac{1}{\theta} \int_{-\theta}^0 \left[f(s)e^{\eta s} + \int_s^0 \frac{d}{dr} (f(r)e^{\eta r}) dr \right] ds \right\|_F \\ &\leq \frac{1}{\theta} \int_{-\theta}^0 \|f(s)\|_F e^{\eta s} ds + \frac{1}{\theta} \left\| \int_{-\theta}^0 \int_s^0 \frac{d}{dr} (f(r)e^{\eta r}) dr ds \right\|_F \\ &\leq \frac{1}{\theta} \|f\|_E + \frac{1}{\theta} \int_{-\theta}^0 \int_s^0 \|f'(r)\|_F e^{\eta r} dr ds \\ &\quad + \frac{1}{\theta} \int_{-\theta}^0 \int_s^0 \|f(r)\|_F \eta e^{\eta r} dr ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\theta} \|f\|_E + \frac{1}{\theta} \int_{-\theta}^0 \|f'\|_E \, ds + \frac{\eta}{\theta} \int_{-\theta}^0 \|f\|_E \, ds \\
&\leq \frac{1}{\theta} \|f\|_E + \|f'\|_E + \eta \|f\|_E \quad \text{for every } \theta > 0.
\end{aligned}$$

Thus $\|f(0)\|_F \leq \eta \|f\|_E + \|f'\|_E$ and hence $\Phi: f \mapsto f(0)$ is continuous from $(D(A), \|\cdot\|_A)$ into F . \square

This result allows us to identify generators of translation semigroups with first derivatives and it is an essential tool in the proof of Proposition 3.10, where we will determine the modulus semigroup of some special translation semigroups.

2. A SPECIAL TYPE OF TRANSLATION SEMIGROUP

If we reformulate Proposition 1.4 we can say that generators of linear translation semigroups on $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ are of the form $Af = f'$ with $D(A) \subseteq \{f \in W^{1,1}: f(0) = \Phi(f)\} =: D_\Phi$, where Φ is a bounded linear operator from $(D(A), \|\cdot\|_A)$ into F . It is obvious that not all operators A with $Af = f'$ and $D(A) = D_\Phi$ are generators. If we choose for example $\Phi = \delta_0$, then $f(0) = \Phi(f)$ is trivially satisfied for any f , hence $D_\Phi = W^{1,1}$, and thus (A, D_Φ) is not a generator. Indeed $D_\Phi \supset \{f \in W^{1,1}: f(0) = 0\}$ implies that $\lambda - A: D_\Phi \rightarrow E$ is not injective for any $\lambda > 0$.

There are at least two “types” of translation semigroups on L^1 which show different properties. We want to give two simple examples and then concentrate on one of these examples which we will study in a more general setting.

Example 2.1. Let $E = L^1[-1, 0]$ and consider the operator $Af = f'$ with domain $D(A) = D_\Phi$, where $\Phi: D(A) \rightarrow F$ is given by $\Phi f = f(-1)$. Then $D(A) = \{f \in W^{1,1}: f(0) = f(-1)\}$. It is well known that A is a generator of the “rotation semigroup” on $E = L^1[-1, 0]$ (see [32, Example A-I, 2.5] for details).

Example 2.2. Let $E = L^1[-1, 0]$ and consider the operator $Af = f'$ with domain $D(A) = D_\Phi$, where $\Phi: E \rightarrow F$ is given by $\Phi f = \int_{-1}^0 b(s)f(s) \, ds$ ($b \in L^\infty[-1, 0]$). Then $D(A) = \{f \in W^{1,1}: f(0) = \int_{-1}^0 b(s)f(s) \, ds\}$. This operator is studied for example in [34 and 42]. It is easy to show that A is the generator of a strongly continuous linear semigroup $(T(t))_{t \geq 0}$. This semigroup plays an important role in the study of age-dependent population equations.

In this paper we are concerned with describing semigroups of the second type. To do so we make the following standard assumptions which are valid in this or a more restrictive version throughout the rest of the paper and we will use this notation frequently without explicitly referring back.

Let F be a Banach space and let $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ be the space of Bochner integrable functions with values in F with norm

$$\|f\|_E = \int_{-\infty}^0 \|f(s)\|_F e^{\eta s} ds$$

for $f \in E$. For further properties of the space E we refer to [12 and 29].

Let Φ be a Lipschitz continuous operator from E into F . We define an unbounded operator A_Φ from E into E by

$$A_\Phi f = f'$$

with domain

$$D(A_\Phi) = \{f \in W^{1,1}((-\infty, 0], F; e^{\eta s} ds) : f(0) = \Phi(f)\},$$

where $W^{1,1} = W^{1,1}((-\infty, 0], F; e^{\eta s} ds) := \{f \in E : f \text{ is absolutely continuous and almost everywhere differentiable with } f' \in E\}$. (In the case where F is reflexive, $W^{1,1} = \{f \in E : f \text{ is absolutely continuous}\}$.)

We will show that there exists a strongly continuous translation semigroup $(T_\Phi(t))_{t \geq 0}$ such that

$$(2) \quad T_\Phi(t)f = \lim_{n \rightarrow \infty} \left[\text{Id} - \frac{t}{n} A_\Phi \right]^{-n} f \quad \text{for } f \in E,$$

i.e. A_Φ generates the semigroup $(T_\Phi(t))_{t \geq 0}$ in the sense of Crandall-Liggett (see [16]). This result is equivalent to an existence and uniqueness statement for the corresponding abstract Cauchy problem. For linear operators Φ (thus if the semigroup $(T_\Phi(t))_{t \geq 0}$ is linear) the Cauchy problem $\frac{d}{dt}u(t) = A_\Phi u(t)$, $u(0) = f \in D(A_\Phi)$ has a unique strong solution u given by $u(t) = T_\Phi(t)f$. In the case of a nonlinear operator Φ (thus if the semigroup $(T_\Phi(t))_{t \geq 0}$ is nonlinear) one obtains mild solutions of the corresponding Cauchy problem for all $f \in E$ (see for example [15]).

After studying this existence result, in the subsequent section we will examine properties such as positivity, irreducibility and compactness of the semigroup $(T_\Phi(t))_{t \geq 0}$ in the case where Φ is linear. Furthermore we will study spectral properties of the semigroup $(T_\Phi(t))_{t \geq 0}$ in order to deduce results on the asymptotic behavior.

Finally we will apply a result of linearized stability by W. Desch and W. Schappacher for the semigroup $(T_\Phi(t))_{t \geq 0}$ under the assumption that Φ is Fréchet-differentiable in a neighborhood of an equilibrium point.

The following existence result is similarly given by R. Vilella-Bressan in [40]. The underlying Banach space considered there is $L^1([-r, 0], F)$ where $r < \infty$.

Theorem 2.3. *Let F be a Banach space, $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ and let $\Phi: E \rightarrow F$ be a Lipschitz continuous operator. The operator A_Φ is a generator of a (nonlinear) semigroup $(T_\Phi(t))_{t \geq 0}$ on E in the sense of Crandall-Liggett. More precisely, the operators $T_\Phi(t)$ given by $T_\Phi(t)f = \lim_{n \rightarrow \infty} [\text{Id} - \frac{t}{n} A_\Phi]^{-n} f$ for all*

$f \in E$ define a strongly continuous semigroup such that $\|T_\Phi(t)f - T_\Phi(t)g\|_E \leq e^{wt} \|f - g\|_E$ where $w = \|\Phi\|_{\text{Lip}}$. (Here $\|\Phi\|_{\text{Lip}}$ denotes the Lipschitz-constant of Φ .)

Proof. In order to apply the theorem of Crandall-Liggett it suffices to show

- (a) there exists a constant w such that $A_\Phi - w$ is dissipative;
- (b) $\text{im}(\text{Id} - \alpha A) = E$ for small $\alpha > 0$; and
- (c) A is densely defined.

(a) To show that there exists w such that $A_\Phi - w$ is dissipative we use the concept of subdifferentials (see for example [32, Section A-II, 2]).

Let N_G (alternatively to $\|\cdot\|_G$) denote the norm of a Banach space G . The set

$$dN_G(f) = \{\varphi \in G' : \|\varphi\|_{G'} \leq 1, \langle f, \varphi \rangle = \|f\|_G\}$$

is called the *subdifferential* of the norm N_G in f . By the theorem of Hahn-Banach $dN_G(f) \neq \emptyset$.

In our situation we are interested in the sets $dN_G(f)$ where either G is the Banach space $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ or G is equal to F .

Lemma 1. *Let F be a Banach space, $E = L^1((-\infty, 0], F; e^{\eta s} ds)$. If $\varphi \in dN_E(f)$ for $f \in E$, then $\varphi(s) \in dN_F(f(s))$ for almost every $s \in (-\infty, 0]$.*

Proof. It is known that $E' = \overline{L^\infty}((-\infty, 0], F', e^{\eta s} ds) = \{f' : (-\infty, 0] \rightarrow F' : f \text{ is } \sigma(F', F) \text{ measurable and } s \mapsto e^{\lambda s} f(s) \text{ is a.e. uniformly bounded}\}$.

Thus $L^\infty((-\infty, 0], F', e^{\eta s} ds) \subseteq \overline{L^\infty}((-\infty, 0], F', e^{\eta s} ds)$ where equality holds for example if F is reflexive. (For a detailed discussion see for example [35, Chapter IV.7].)

Let $\varphi \in dN_E(f)$, i.e., $\varphi \in E' = L^1[(-\infty, 0], F', e^{\eta s} ds]'$, $\langle f, \varphi \rangle = \|f\|$ and $\|\varphi\| \leq 1$. Hence φ is bounded and moreover $\|\varphi(s)\|_{F'} \leq 1$ for almost every $s \in (-\infty, 0]$. Thus

$$\begin{aligned} \int_{-\infty}^0 \|f(s)\|_F e^{\eta s} ds &= \|f\|_E = \int_{-\infty}^0 \langle f(s), \varphi(s) \rangle e^{\eta s} ds \\ &\leq \int_{-\infty}^0 \|f(s)\|_F \cdot \|\varphi(s)\|_{F'} e^{\eta s} ds \\ &\leq \int_{-\infty}^0 \|f(s)\|_F e^{\eta s} ds \cdot \|\varphi\|_{E'} \\ &\leq \int_{-\infty}^0 \|f(s)\|_F e^{\eta s} ds. \end{aligned}$$

Hence

$$\int_{-\infty}^0 \langle f(s), \varphi(s) \rangle e^{\eta s} ds = \int_{-\infty}^0 \|f(s)\|_F e^{\eta s} ds.$$

Since $\|\varphi(s)\| \leq 1$ we have $\|f(s)\| - \langle f(s), \varphi(s) \rangle \geq \|f(s)\| - \|f(s)\| = 0$ for almost every $s \in (-\infty, 0]$. Hence $\int_{-r}^0 [\|f(s)\| - \langle f(s), \varphi(s) \rangle] e^{\eta s} ds = 0$ implies $\|f(s)\| = \langle f(s), \varphi(s) \rangle$ for almost every $s \in (-\infty, 0]$.

Altogether we consequently have $\varphi(s) \in F'$, $\|\varphi(s)\| \leq 1$ and $\langle f(s), \varphi(s) \rangle = \|f(s)\|$ for almost every $s \in (-\infty, 0]$ which is equivalent to $\varphi(s) \in dN_F(f(s))$ for almost every $s \in (-\infty, 0]$. \square

The following result will be needed in the proof of the dissipativeness of $A_\Phi - w$.

Lemma 2. *Let $f \in L^1((-\infty, 0], F; e^{\eta s} ds)$ be an absolutely continuous function from $(-\infty, 0]$ into F and let $\varphi \in dN_E(f)$. Then $\|f(\cdot)\|$ is almost everywhere differentiable and $\frac{d}{ds} \|f(s)\| = \langle f'(s), \varphi(s) \rangle$ almost everywhere.*

The proof of this lemma is a direct consequence of [24, Lemma 1.3].

Proof of Proposition 2.3 continued. (a) To show dissipativeness of $A_\Phi - w$ for a constant w we have to show that for all $f, g \in D(A_\Phi)$ and all $\psi \in dN_E(f - g)$ we have $\langle A_\Phi f - A_\Phi g, \psi \rangle \leq w \|f - g\|$. Thus let $f, g \in D(A_\Phi)$ and let $\psi \in dN_E(f - g)$. By Lemma 1 we have $\psi(s) \in dN_F(f(s) - g(s))$ for almost every s and thus

$$\begin{aligned} \langle A_\Phi f - A_\Phi g, \psi \rangle &= \int_{-\infty}^0 \langle (f - g)'(s), \psi(s) \rangle e^{\eta s} ds \\ &= \int_{-\infty}^0 \left[\frac{d}{ds} \|f(s) - g(s)\| \right] e^{\eta s} ds \end{aligned}$$

where the second equality holds by Lemma 2 since $f - g$ is absolutely continuous. Thus we obtain:

$$\begin{aligned} \langle A_\Phi f - A_\Phi g, \psi \rangle &= \|f(0) - g(0)\| - \eta \int_{-\infty}^0 \|f(s) - g(s)\| e^{\eta s} ds \\ &\leq \|\Phi f - \Phi g\| \quad \text{since } f, g \in D(A_\Phi) \text{ and } \eta \geq 0. \end{aligned}$$

Thus $\langle A_\Phi f - A_\Phi g, \psi \rangle \leq \|\Phi\|_{\text{Lip}} \cdot \|f - g\|$ which proves that $A_\Phi - w$ is dissipative if we choose $w \geq \|\Phi\|_{\text{Lip}}$.

(b) Let $g \in E$. We have to show that there exists $f \in D(A_\Phi)$ such that $f - \alpha A_\Phi f = g$ for small $\alpha > 0$. The identity $f - \alpha A_\Phi f = g$ is formally equivalent to $f' = \frac{1}{\alpha} f - \frac{1}{\alpha} g$.

Under the assumption $f \in D(A_\Phi)$ this ordinary differential equation yields ($t \leq 0$):

$$(3) \quad f(t) = e^{t/\alpha} \Phi(f) + \frac{1}{\alpha} \int_t^0 e^{(t-s)/\alpha} g(s) ds.$$

Let $V_\alpha : E \rightarrow E$ be defined by $V_\alpha f(t) := e^{t/\alpha} \Phi(f) + \frac{1}{\alpha} \int_t^0 e^{(t-s)/\alpha} g(s) ds$. Let $f, h \in E$. Then

$$\begin{aligned} \|V_\alpha f - V_\alpha h\| &= \int_{-\infty}^0 \|V_\alpha f(t) - V_\alpha h(t)\| e^{\eta t} dt \\ &= \int_{-\infty}^0 e^{t/\alpha} \|\Phi(f) - \Phi(h)\| e^{\eta t} dt \end{aligned}$$

$$= \frac{\alpha}{1 + \alpha\eta} \cdot \|\Phi\|_{\text{Lip}} \cdot \|f - h\|.$$

This shows that V_α is a strict contraction if α is small enough. By the fixed point theorem of Banach, there exists a unique fixed point \bar{f} with $V_\alpha \bar{f} = \bar{f}$. Hence $\bar{f}(0) = V_\alpha \bar{f}(0) = \Phi(\bar{f})$, and thus $\bar{f} \in D(A_\Phi)$.

By the construction preceding formula (3), $(\text{Id} - \alpha A_\Phi)\bar{f} = g$ and hence $\bar{f} = (\text{Id} - \alpha A_\Phi)^{-1}g$.

For abbreviation we define operators $J_\alpha^\Phi g := \bar{f} = (\text{Id} - \alpha A_\Phi)^{-1}g$ in analogy to the definition of the resolvent operator in the case of linear operators. (More precisely, if Φ is linear $J_\alpha^\Phi = \frac{1}{\alpha}R(\frac{1}{\alpha}, A_\Phi)$.)

(c) It remains to show that $D(A_\Phi)$ is dense in L^1 . It is easy to show that A_0 is a linear operator and that

$$J_\alpha^0 g(t) = \frac{1}{\alpha} \int_t^0 e^{(t-s)/\alpha} g(s) ds \quad \text{for } g \in E.$$

Moreover A_0 is the generator of a semigroup $(T_0(t))_{t \geq 0}$. (The explicit formula for $(T_0(t))_{t \geq 0}$ is formula (1) in the introduction.) In particular A_0 is densely defined and thus $J_\alpha^0 g \rightarrow g$ for all $g \in E$ if $\alpha \rightarrow 0$. Let $g \in E$. We will show that $\lim_{\alpha \rightarrow 0} J_\alpha^\Phi g = g$.

Using the computations of part (b) we obtain

$$g(t) - (J_\alpha^\Phi g)(t) = g(t) - e^{t/\alpha} (J_\alpha^\Phi g)(0) - J_\alpha^0 g(t).$$

If we denote by ε_λ the function $\varepsilon_\lambda(t) = e^{\lambda t}$, then

$$\begin{aligned} \|g - J_\alpha^\Phi g\|_E &\leq \|\varepsilon_{1/\alpha} \cdot J_\alpha^\Phi g(0)\|_E + \|g - J_\alpha^0 g\|_E \\ &\leq \frac{\alpha}{1 + \alpha\eta} \|\Phi(J_\alpha^\Phi g)\|_F + \|g - J_\alpha^0 g\|_E \\ &\leq \frac{\alpha}{1 + \alpha\eta} [\|\Phi\|_{\text{Lip}} \cdot \|J_\alpha^\Phi g - g\|_E + \|\Phi(g)\|_F] + \|g - J_\alpha^0 g\|_E. \end{aligned}$$

Hence

$$\|g - J_\alpha^\Phi g\|_E \cdot \left[1 - \frac{\alpha}{1 + \alpha\eta} \|\Phi\|_{\text{Lip}}\right] \leq \frac{\alpha}{1 + \alpha\eta} \|\Phi(g)\|_F + \|g - J_\alpha^0 g\|_E,$$

or equivalently

$$\|g - J_\alpha^\Phi g\|_E \leq \frac{1}{1 - \frac{\alpha}{1 + \alpha\eta} \|\Phi\|_{\text{Lip}}} \cdot \left[\frac{\alpha}{1 + \alpha\eta} \|\Phi(g)\|_F + \|g - J_\alpha^0 g\|_E \right].$$

As we remarked before $\lim_{\alpha \rightarrow 0} J_\alpha^0 g = g$. Together with $\lim_{\alpha \rightarrow 0} \frac{\alpha}{1 + \alpha\eta} = 0$ this yields $\lim_{\alpha \rightarrow 0} \|g - J_\alpha^\Phi g\|_E = 0$. \square

Corollary 2.4. *The semigroup $(T_\Phi(t))_{t \geq 0}$ is a translation semigroup. Furthermore $(T_\Phi(t))_{t \geq 0}$ satisfies the following identity: Let $g \in E$, then*

$$(4) \quad [T_\Phi(t)g](x) = \begin{cases} g(x+t) & \text{if } x+t < 0, \\ \Phi(T_\Phi(x+t)g) & \text{if } x+t \geq 0, \end{cases}$$

$$\begin{aligned}
(5) \quad &= [T_0(t)]g(x) + \begin{cases} 0 & \text{if } x+t < 0, \\ \Phi(T_\Phi(x+t)g) & \text{if } x+t \geq 0, \end{cases} \\
&= [T_0(t)]g(x) + \begin{cases} 0 & \text{if } x+t < 0, \\ \Phi(g) & \text{if } x+t = 0, \\ [T_\Phi(x+t)g](0) & \text{if } x+t > 0, \end{cases} \\
(6) \quad &=: [T_0(t)g](x) + [S(t)g](x),
\end{aligned}$$

where the operator family $(S(t))_{t \geq 0}$ is defined by the last identity.

Remark. We call formula (4) (resp. (5)) the “translation property” of $(T(t))_{t \geq 0}$ and will use this property frequently in the sequel.

Sketch of the proof. Solving the Cauchy problem corresponding to $(T_\Phi(t))_{t \geq 0}$ by integration along characteristics one can show that there exists a family of Lipschitz continuous operators $(\tilde{T}(t))_{t \geq 0}$ on E which satisfies

$$[\tilde{T}(t)g](x) = \begin{cases} g(x+t) & \text{if } x+t < 0, \\ \Phi(\tilde{T}(x+t)g) & \text{if } x+t \geq 0, \end{cases} \quad \text{for all } g \in E,$$

such that additionally $\tilde{T}(t)g = T_\Phi(t)g$ for all $g \in D(A_\Phi)$. (See [18] for details.) By the continuity of $T_\Phi(t)$ one thus gets $T_\Phi(t)g = \tilde{T}(t)g$ for all $g \in E$ and hence

$$\begin{aligned}
[T_\Phi(t)g](x) &= [\tilde{T}(t)g](x) = \begin{cases} g(x+t) & \text{if } x+t < 0, \\ \Phi(\tilde{T}(x+t)g) & \text{if } x+t \geq 0, \end{cases} \\
&= \begin{cases} g(x+t) & \text{if } x+t < 0, \\ \Phi(T_\Phi(x+t)g) & \text{if } x+t \geq 0, \end{cases} \quad \text{for all } g \in E.
\end{aligned}$$

Let $t > 0$ be fixed. For $x, x' \geq -t$ we have

$$\begin{aligned}
\|T_\Phi(t)g(x) - T_\Phi(t)g(x')\|_F &= \|\Phi(T_\Phi(t+x)g) - \Phi(T_\Phi(t+x')g)\|_F \\
&\leq \|\Phi\|_{\text{Lip}} \cdot \|T_\Phi(t+x)g - T_\Phi(t+x')g\|_E.
\end{aligned}$$

By the strong continuity of $(T_\Phi(t))_{t \geq 0}$ continuity of the map $x \mapsto T(t)g(x)$ from $[-t, 0]$ into F follows. In particular we have $\Phi(T(t+x)g) = \Phi(g)$ if $t+x = 0$ and $T_\Phi(x+t)g(0) = \Phi(T_\Phi(x+t)g)$ if $x+t > 0$. Thus the representation in formula (5) follows. \square

The translation property (4) and the decomposition (6) for the semigroup $(T_\Phi(t))_{t \geq 0}$ is essential for the study of properties of $(T_\Phi(t))_{t \geq 0}$ in terms of Φ .

3. SOME PROPERTIES OF $(T_\Phi(t))_{t \geq 0}$

We adopt the notation of the last section. Consider the semigroup $(T_\Phi(t))_{t \geq 0}$ on the Banach space

$$E = L^1((-\infty, 0], F; e^{\eta s} ds)$$

with the corresponding generator A_Φ . The discussion of some important properties of the semigroup $(T_\Phi(t))_{t \geq 0}$ is led by the idea that properties of Φ may be passed onto the semigroup via the representation in formula (6).

We start our discussion for the case where Φ and thus $(T_\Phi(t))_{t \geq 0}$ is linear. Corresponding results for the nonlinear case (if applicable) are also stated and should be seen as independently but similarly obtained results.

We will consider the following properties:

- (a) Compactness;
- (b) Positivity;
- (c) Irreducibility;
- (d) Domination;
- (e) Localization of the spectrum;
- (f) Linearization in the nonlinear case.

In this section we assume that $\Phi \in \mathcal{L}(E, F)$ is a linear bounded operator if not stated otherwise.

For $\lambda > -\eta$ we define a family of operators Φ_λ on F which is induced by Φ in the following way. For $x \in F$ we define $\Phi_\lambda x := \Phi(e^{\lambda \cdot} x)$. For convenience we denote the map $t \mapsto e^{\lambda t} x$ by $\varepsilon_\lambda \otimes x$ ($x \in F$), thus $\Phi_\lambda x = \Phi(\varepsilon_\lambda \otimes x)$.

It is easy to verify that $\lambda > -\eta$ implies that $\varepsilon_\lambda \otimes x \in E$ and thus $\Phi_\lambda \in \mathcal{L}(F)$. Furthermore since Φ is a linear operator from E into F the resolvent of A_Φ is given by

$$R(\lambda, A_\Phi)f = R(\lambda, A_0)f + \varepsilon_\lambda \otimes R(1, \Phi_\lambda)\Phi R(\lambda, A_0)f$$

for $f \in E$ and large λ , where

$$R(\lambda, A_0)f(t) = \int_t^0 e^{\lambda(t-s)} f(s) ds$$

for $f \in E$, $\lambda > -\eta$ and $t \in (-\infty, 0]$. Indeed, $\lambda f - A_\Phi f = g$ is equivalent to $f(t) = e^{\lambda t} f(0) + R(\lambda, A_0)g(t)$. Since $f \in D(A_\Phi)$ we have $f(0) = \Phi(f) = \Phi_\lambda f(0) + \Phi R(\lambda, A_0)g$. Thus we obtain that $f(0) = R(1, \Phi_\lambda)\Phi R(\lambda, A_0)g$ for large λ , since $R(1, \Phi_\lambda)$ exists for large λ . Hence $R(\lambda, A_\Phi)g = f$ is of the given form (compare also formula (3)).

(a) Compactness. Let $\Phi \in \mathcal{L}(E, F)$ be a compact operator. It is evidently not true that $(T_\Phi(t))_{t \geq 0}$ is eventually compact (i.e. $T_\Phi(t)$ compact for large t). For fixed t and $f \in E$ we have by formula (4) that $T_\Phi(t)f(x) = f(x+t)$ for x with $x \leq -t$. Thus the operator $T_\Phi(t)$ is not compact, even if Φ is compact. By the same reason one can also not expect eventually weak compactness of the

semigroup under the assumption that Φ is weakly compact. (This argument does not hold in the Banach space $E = L^1([-r, 0], F; e^{\eta s} ds)$ with $r < \infty$.) Thus instead of eventual compactness we will study under which conditions on Φ the semigroup $(T_\Phi(t))_{t \geq 0}$ has a weaker compactness property. For this we introduce the notion of essential compactness. (See also [32, Section C-IV, 2] where the strongly related concept of quasi-compactness is used: Indeed, if a semigroup $(T(t))_{t \geq 0}$ is essentially compact, then the rescaled semigroup given by $\hat{T}(t) := e^{-\omega(T(t)) \cdot t} T(t)$ is for $t \geq 0$ quasi-compact.)

Definition 3.1. *A semigroup $(T_\Phi(t))_{t \geq 0}$ is called essentially compact (resp. essentially weakly compact) if $T(t) = R(t) + S(t)$ for all t , where $(R(t))_{t \geq 0}$ is a strongly continuous semigroup, the operators $S(t)$ are compact (resp. weakly compact) and furthermore $\omega(R(t)) < \omega(T(t))$.*

Recall that $\omega(T(t)) = \omega(A) = \inf\{\omega: \|T(t)\| \leq Me^{\omega t}, M \geq 0\}$, and $s(A) = \sup\{\operatorname{Re} \lambda: \lambda \in \sigma(A)\}$ if $\sigma(A) \neq \emptyset$ and $s(A) = -\infty$ if $\sigma(A) = \emptyset$. The constant $\omega(A)$ determines the growth of the semigroup $(T(t))_{t \geq 0}$. In general $s(A) \leq \omega(A)$, but in certain cases equality holds. This and further notation concerning the spectral properties of bounded linear operators and strongly continuous linear semigroups can be found in [32 and 21] for example.

Essential compactness is a very useful tool in determining asymptotic behavior of semigroups. It implies that $\omega(T(t)) = s(A)$, where A is the generator of $(T(t))_{t \geq 0}$. Furthermore, under the assumption that the semigroup $(T(t))_{t \geq 0}$ is essentially compact, the boundary spectrum $\sigma_0(A) := \{\lambda: \operatorname{Re} \lambda = s(A)\}$ consists of poles of the resolvent of A with finite dimensional residue. The same is true for essentially weakly compact semigroups, if the underlying Banach space possesses the Dunford-Pettis property (thus for example for L^1 and $C(K)$ spaces). See [34, 35 and 42] for details.

For the semigroup $(T_\Phi(t))_{t \geq 0}$ of concern we obtain the following result:

Proposition 3.2. *Let $\Phi \in \mathcal{L}(E, F)$ such that $\Phi \circ A_0$ has a continuous extension from E to F (which we denote by $[\Phi \circ A_0]$). Furthermore let Φ be compact (resp. weakly compact) and let $s(A_\Phi) > -\eta$. Then the semigroup $(T_\Phi(t))_{t \geq 0}$ is essentially compact (resp. essentially weakly compact).*

Proof. We use the following variant of the theorem of Arzelà-Ascoli.

Lemma. *Let F be a Banach space. A subset $B \subseteq C([-r, 0], F)$ is compact (resp. relatively weakly compact) if*

- (a) B is equibounded,
- (b) B is equicontinuous,
- (c) $\{f(x): f \in B\}$ is relatively compact (resp. relatively weakly compact).

By formula (4) (resp. (6)) we know that $T_\Phi(t) = T_0(t) + S(t)$ for every t , where

$$[S(t)f](x) = \begin{cases} \Phi(T_\Phi(x+t)f) & \text{if } x+t \geq 0, \\ 0 & \text{if } x+t < 0. \end{cases}$$

Let $t > 0$ be fixed. Let U be the unit ball of $L^1((-\infty, 0], F; e^{\eta s} ds)$. We show that $S(t)U$ is relatively compact (relatively weakly compact) in $C([-t, 0], F)$ and thus in $L^1((-\infty, 0], F; e^{\eta s} ds)$, hence that $S(t)$ is compact.

For $x \geq -t$ we have $[S(t)f](x) = \Phi(T_\Phi(x+t)f)$. Thus

$$\begin{aligned} \sup_{x \in [-t, 0]} \|[S(t)f](x)\|_F &\leq \sup_{x \in [-t, 0]} \|\Phi\| \cdot \|T_\Phi(x+t)f\|_E \\ &\leq \sup_{x \in [-t, 0]} \|\Phi\| \cdot e^{\|\Phi\|(x+t)} \cdot \|f\|_E \\ &= \|\Phi\| \cdot e^{\|\Phi\|t} \cdot \|f\|_E \leq \|\Phi\| \cdot e^{\|\Phi\|t} \quad \text{uniformly for } f \in U. \end{aligned}$$

This proves the equiboundedness. To show the equicontinuity we observe that

$$\begin{aligned} [S(t)f](x) &= \Phi(T_\Phi(x+t)f - f) + \Phi(f) = \Phi\left(A_\Phi \int_0^{x+t} T_\Phi(s)f ds\right) + \Phi(f) \\ &= [\Phi \circ A_\Phi] \left(\int_0^{x+t} T_\Phi(s)f ds\right) + \Phi(f). \end{aligned}$$

Thus we obtain

$$\|[S(t)f](x) - [S(t)f](y)\|_F = \left\| \Phi \circ A_\Phi \left(\int_y^{x+t} T_\Phi(s)f ds \right) \right\|_F.$$

Now $[\Phi \circ A_0]$ is bounded by assumption. One can easily see that this is true if and only if $\Phi \circ A_\Phi$ has a continuous extension $[\Phi \circ A_\Phi]$ from E to F . (For a detailed discussion of this assumption and its consequences see [19].) Thus

$$\|[S(t)f](x) - [S(t)f](y)\|_F \leq \|[\Phi \circ A_\Phi]\| \cdot |x-y| \cdot e^{\|\Phi\|t} \cdot \|f\|_E \leq M \cdot |x-y| \cdot \|f\|_E$$

which converges to 0 if $|x-y| \rightarrow 0$, uniformly for $f \in U$. This proves the equicontinuity of $S(t)U$.

It remains to show that $\{[S(t)f](x) : f \in U\}$ is relatively compact (relatively weakly compact) in F for any $x \in [-t, 0]$. But this follows directly from the corresponding compactness assumption on Φ .

Altogether we know that $T_\Phi(t) = T_0(t) + S(t)$ for $t \geq 0$, where $(T_0(t))_{t \geq 0}$ is a strongly continuous linear semigroup with $\omega(T_0(t)) = -\eta$ and $S(t)$ is compact (resp. weakly compact). By assumption $s(A_\Phi) > -\eta$, thus $\omega(T_\Phi(t)) \geq s(A_\Phi) > -\eta = \omega(T_0(t))$ which implies that $(T_\Phi(t))_{t \geq 0}$ is essentially compact (resp. essentially weakly compact). \square

Remark 3.3. If we consider the Banach space $E = L^1([-r, 0], F; e^{\eta s} ds)$ with $r < \infty$, then compactness of Φ implies eventually compactness of $(T_\Phi(t))_{t \geq 0}$ and, analogously, weak compactness of Φ implies eventually weak compactness of $(T_\Phi(t))_{t \geq 0}$. This can be proved using the same argument as in the proof of the proposition.

(b) Positivity. Let F be a Banach lattice. Under the canonical order induced by the order of F the Banach space $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ is a Banach lattice as well (see [35]). The positivity of the semigroup $(T_\Phi(t))_{t \geq 0}$ can be described in terms of Φ and vice versa.

Proposition 3.4. *Let F be a Banach lattice and $\Phi \in \mathcal{L}(E, F)$. The following assertions are equivalent:*

- (i) Φ is positive, i.e. $\Phi f \geq 0$ if $f \geq 0$.
- (ii) The semigroup $(T_\Phi(t))_{t \geq 0}$ is positive.
- (iii) The resolvent $R(\lambda, A_\Phi)$ is positive for large λ .

Proof. The equivalence of (ii) and (iii) is well known (see e.g. [32, Proposition B-II, 1.1]).

“(i) \Rightarrow (iii)” Since $R(\lambda, A_\Phi) = R(\lambda, A_0) + \varepsilon_\lambda \otimes R(1, \Phi_\lambda)\Phi R(\lambda, A_0)$ the positivity of $R(\lambda, A_\Phi)$ follows by the positivity of the operators $R(\lambda, A_0)$, Φ and $R(1, \Phi_\lambda)$.

“(ii) \Rightarrow (i)” Let $(T_\Phi(t))_{t \geq 0}$ be positive and assume Φ is not positive. Then there exists $f \in E_+$ such that $\Phi f \notin F_+$, in particular there exists $x' \in F'_+$, $x' \neq 0$ such that $\langle \Phi f, x' \rangle = -1$.

Let $f_n := \frac{1}{n} \int_0^{1/n} T_\Phi(t)f \, ds$ ($n \in \mathbb{N}$). Then $f_n \in D(A_\Phi)$ and $f_n \rightarrow f$. Furthermore $f \in E_+$ implies that $f_n \in E_+$ for large n . Since $f_n \in D(A_\Phi)$ we thus have $\Phi(f_n) = f_n(0) \in F_+$. On the other hand $\langle \Phi f_n, x' \rangle \rightarrow \langle \Phi f, x' \rangle = -1$ which is a contradiction to the positivity of $\Phi(f_n)$. \square

Again we have a nonlinear version.

Proposition 3.5. *Let Φ be a positive Lipschitz continuous (nonlinear) map from E to F . Then $(T_\Phi(t))_{t \geq 0}$ is a positive semigroup.*

The proof follows similarly to the first part of the preceding proposition, by showing the positivity of the operators J_α^Φ because the operators $T_\Phi(t)$ are given by the relation $T_\Phi(t) = \lim_{n \rightarrow \infty} [J_{t/n}^\Phi]^n$ (see formula (2)). Indeed, for positive functions $f \in E$ and $\alpha > 0$ we showed that $J_\alpha^\Phi f$ is the fixed point of the positive operator V_α , thus positive.

(c) **Irreducibility.** A strong positivity property for a semigroup is irreducibility. It assures the simplicity of poles of the resolvent of generators A (see [32, Proposition C-III, 3.5]) and is thus important in the study of asymptotic behavior of semigroups. For a discussion of irreducibility, quasi-interior points and lattice-ideals we refer the reader to the book of H. H. Schaefer [35, Chapters II.2, II.6 and II.8].

Definition 3.6. *A linear semigroup $(T(t))_{t \geq 0}$ on a Banach lattice E is called irreducible, if for all $E \ni f > 0$ ($f \in E_+$, $f \neq 0$) and all $E' \ni \varphi > 0$ ($\varphi \in E'_+$, $\varphi \neq 0$) there exists $t > 0$ such that $\langle T(t)f, \varphi \rangle > 0$. A bounded operator T on a Banach lattice is called irreducible, if the semigroup $\{T^n : n \in \mathbb{N}\}$ is irreducible.*

Remark. A very illustrative equivalent definition for irreducibility of a semigroup is that there exists no nontrivial closed $T(t)$ -invariant lattice ideal. If the Banach lattice E has order continuous norm (in particular if $E = L^p$) this

means that one cannot decompose the E such that $E = E_1 \oplus E_2$ where E_1 and E_2 are proper closed ideals in E , and that $T(t)$ is of the form

$$\begin{pmatrix} T_{11}(t) & T_{12}(t) \\ 0 & T_{22}(t) \end{pmatrix}$$

(see also [32, Definition C-III, 3.1]).

Proposition 3.7. *Let F be a Banach lattice and let $\Phi \in \mathcal{L}(E, F)$ be strictly positive (i.e. $f > 0$ implies $\Phi f > 0$). Furthermore let Φ_λ be irreducible for some $\lambda > -\eta$. Then $(T_\Phi(t))_{t \geq 0}$ is irreducible.*

For the proof we need the following two lemmas.

Lemma 3. *Let E be a Banach lattice and $(T(t))_{t \geq 0}$ a strongly continuous semigroup with generator A . The following assertions are equivalent:*

- (i) $(T(t))_{t \geq 0}$ is irreducible.
- (ii) For some (every) $\lambda > s(A)$ the resolvent $R(\lambda, A)$ is irreducible.
- (iii) For some (every) $\lambda > s(A)$ we have $R(\lambda, A)f$ is a quasi-interior point of E_+ whenever $f > 0$ (i.e. for all $E' \ni \varphi > 0$ we have $\langle R(\lambda, A)f, \varphi \rangle > 0$).

A proof of this result can be found for example in [32, C-III, 3.1].

Let us now consider our special situation again. The following statement is true:

Lemma 4. *Let \hat{x} be a quasi-interior point of F_+ . Then $\varepsilon_\lambda \otimes \hat{x}$ is a quasi-interior point of $L^1((-\infty, 0], F; e^{\eta s} ds)_+$.*

Proof. Let \hat{x} be a quasi-interior point of F_+ . Hence the ideal $I_{\hat{x}}$ which is generated by \hat{x} is dense in F (see [35, Definition II.6.1] for this equivalent condition). Since ε_λ is a quasi-interior point of $L^1((-\infty, 0]; e^{\eta s} ds)_+$ the ideal I_{ε_λ} is dense in $L^1((-\infty, 0]; e^{\eta s} ds)_+$. Hence $I_{\varepsilon_\lambda} \otimes I_{\hat{x}}$ is dense in $L^1((-\infty, 0]; e^{\eta s} ds) \tilde{\otimes}_\pi F$. But $I_{\varepsilon_\lambda \otimes \hat{x}} \supseteq I_{\varepsilon_\lambda} \otimes I_{\hat{x}}$ implies that $I_{\varepsilon_\lambda \otimes \hat{x}}$ is dense in E and hence that $\varepsilon_\lambda \otimes \hat{x}$ is a quasi-interior point of E_+ . \square

Proof of Proposition 3.7. By Lemma 3 it is sufficient to show that the resolvent $R(\lambda, A_\Phi)$ is irreducible. First we show that $R(1, \Phi_\lambda)\Phi R(\lambda, A_0)f$ is a quasi-interior point of F_+ for large λ whenever $E \ni f > 0$. For $f > 0$ the explicit formula for $R(\lambda, A_0)$ shows that $R(\lambda, A_0)f > 0$, hence due to the strict positivity of Φ we have $\Phi R(\lambda, A_0)f > 0$. If we choose λ large such that $r(\Phi_\lambda) < 1$, then the irreducibility of $R(1, \Phi_\lambda)$ follows from the irreducibility of Φ_λ . Altogether we thus have by Lemma 3 that $R(1, \Phi_\lambda)\Phi R(\lambda, A_0)f$ is a quasi-interior point of F_+ . By Lemma 4 this implies that $\varepsilon_\lambda \otimes R(1, \Phi_\lambda)\Phi R(\lambda, A_0)f$ is quasi-interior point of E_+ .

Since $R(\lambda, A_0)$ is positive, $R(\lambda, A_0)f + \varepsilon_\lambda \otimes R(1, \Phi_\lambda)\Phi R(\lambda, A_0)f$ is a quasi-interior point of E_+ as well. Now Lemma 3 implies that $R(\lambda, A_\Phi) = R(\lambda, A_0) + \varepsilon_\lambda \otimes R(1, \Phi_\lambda)\Phi R(\lambda, A_0)$ is an irreducible operator on E . \square

Remark. The assumptions of Proposition 3.7 are sufficient but not necessary for the irreducibility of $(T(t))_{t \geq 0}$. Indeed the assumption that Φ is strictly positive is very strong as the following example shows.

Example. Let $F = \mathbb{R}$, $E = L^1(\mathbb{R}_-)$ and let $\Phi \in \mathcal{L}(E, F)$ be given by $\Phi f = \int_{-\infty}^0 \beta(s) f(s) ds$ where $\beta(s) = 0$ for $s \in (-n-1, -n]$, $n = 0, 2, 4, \dots$, and $\beta(s) = 1$ for $s \in (-n-1, -n]$, $n = 1, 3, 5, \dots$. Then $\Phi_\lambda: \mathbb{R} \rightarrow \mathbb{R}$ is of the form

$$\Phi_\lambda(c) = \int_{-\infty}^0 \beta(s) e^{\lambda s} ds \cdot c = \sum_{k=0}^{\infty} \int_{-2k-2}^{-2k-1} e^{\lambda s} ds \cdot c > 0$$

and thus irreducible, but Φ is not strictly positive since $\Phi f = 0$ for any function f with

$$\text{supp } f \subseteq \bigcup_{k=0}^{\infty} (-2k-1, -2k].$$

On the other hand $(T_\Phi(t))_{t \geq 0}$ is irreducible. This can be verified by considering all possible ideals which are invariant under the left-shift $(T_0(t))_{t \geq 0}$, thus of the form $I_r = \{f \in L^1(\mathbb{R}_-): f(s) = 0 \text{ for almost every } s \in [-r, 0]\}$.

By formulas (4)–(6) we know that for all $f \in E$ and $x \in (-\infty, 0]$

$$\begin{aligned} T_\Phi(t)f(x) &= T_0(t)f(x) + S(t)f(x) \\ &= T_0(t)f(x) + \begin{cases} 0 & \text{if } t+x < 0, \\ \Phi(T_0(t+x)f) & \text{if } t+x \geq 0, \end{cases} \\ &= T_0(t)f(x) + \begin{cases} 0 & \text{if } t+x < 0, \\ \Phi(S(t+x)f) & \text{if } t+x \geq 0. \end{cases} \end{aligned}$$

Let $0 < f \in I_r$, then $f(x) > 0$ for $x \in M$ where $M \subseteq (-\infty, r)$ is a set with positive Lebesgue measure $\mu(M) > 0$. This implies that there exist $t > 0$ and $n \in \{1, 3, 5, \dots\}$ such that $\mu(M_{-t} \cap (-n-1, -n]) > 0$ (where $M_{-t} = \{-t+m: m \in M\}$) and $T_0(t+x)f(z) > 0$ for $z \in M_{-t} \cap (-n-1, -n]$. Thus $\Phi(T_0(t+x)f) > 0$ and hence $T_\Phi(t)f(x) > 0$ which implies the irreducibility of $(T_\Phi(t))_{t \geq 0}$.

This simple example gives the idea for the formulation and the proof of the following very technical proposition which is in applications often more appropriate than Proposition 3.7.

Proposition 3.8. *For every $f > 0$, there exist $\bar{t} > 0$ and $\varepsilon > 0$ such that $\Phi(T_0(t)f)$ is a quasi-interior point of F_+ , whenever $t \in [\bar{t} - \varepsilon, \bar{t} + \varepsilon]$. Then $(T_\Phi(t))_{t \geq 0}$ is irreducible.*

Proof. Let $0 < f \in E$ and $0 < \varphi \in E' = \overline{L^\infty}((-\infty, 0], F'; e^{ns} ds)$ and choose $\bar{t} > 0$ and $\varepsilon > 0$ such that the assertions of the proposition are satisfied.

Since $\varphi > 0$ there exists a set N with $\mu(N) > 0$ such that $\varphi(x) > 0$ for all $x \in N$. Choose $x_0 \in N$ (notice that $-\infty < x_0 < 0$) such that

$$\mu(N \cap [x_0, x_0 + \varepsilon]) > 0.$$

Let $\tilde{t} := \bar{t} - x_0$.

By formula (4) we have for all $t \geq 0$ that

$$T_\Phi(t)f(x) = \begin{cases} f(x+t) & \text{if } x+t < 0, \\ \Phi(T_\Phi(x+t)f) & \text{if } x+t \geq 0. \end{cases}$$

Thus

$$\begin{aligned} \langle T_\Phi(\tilde{t})f, \varphi \rangle &= \int_{-\infty}^0 \langle T_\Phi(\tilde{t})f(x), \varphi(x) \rangle dx \\ &= \int_{-\infty}^{-\tilde{t}} \langle f(x+\tilde{t}), \varphi(x) \rangle dx + \int_{-\tilde{t}}^0 \langle \Phi(T_\Phi(x+\tilde{t})f), \varphi(x) \rangle dx \\ &\geq \int_{-\tilde{t}}^0 \langle \Phi(T_0(x+\tilde{t})f), \varphi(x) \rangle dx + \int_{-\tilde{t}}^0 \langle \Phi(S(x+\tilde{t})f), \varphi(x) \rangle dx \\ &\geq \int_{N \cap [x_0, x_0 + \varepsilon]} \langle \Phi(T_0(x+\tilde{t})f), \varphi(x) \rangle dx, \end{aligned}$$

since $-\tilde{t} = x_0 - \bar{t} \leq x_0$. But $x + \tilde{t} \geq x_0 + \tilde{t} = \bar{t}$ and $x + \tilde{t} \leq x_0 + \varepsilon + \tilde{t} = \bar{t} + \varepsilon$, thus the integrand is positive on a set of measure > 0 , thus $\langle T(\tilde{t})f, \varphi \rangle > 0$. \square

(d) Domination. In this part we want to study a property which is very much related to positivity. In order to find conditions for stability of a given semigroup one tries to compare it with other semigroups, for example, one is interested in semigroups which dominate the original one. A more technical question is, whether one can find a smallest such dominating semigroup.

In general these two questions are answered negatively, neither a dominating nor a smallest dominating semigroup must exist. For examples we refer to [3].

Definition 3.9. Let E be a Banach lattice with order continuous norm. (Banach lattices with order continuous norm are for example L^p ($p < \infty$) and c_0 , see [35, Chapter II.5] for a complete discussion.)

(a) The semigroup $(T(t))_{t \geq 0}$ dominates the semigroup $(S(t))_{t \geq 0}$ if $|S(t)f| \leq T(t)|f|$ for all $f \in E, t \geq 0$.

(b) A semigroup $(T(t))_{t \geq 0}$ dominating $(S(t))_{t \geq 0}$ is called modulus semigroup of $(S(t))_{t \geq 0}$, if $(T(t))_{t \geq 0}$ is dominated by any other semigroup dominating $(S(t))_{t \geq 0}$.

Lemma 5. Let F be an order complete Banach lattice,

$$E = L^1((-\infty, 0], F; e^{\eta s} ds).$$

Furthermore let $\Phi, \Psi \in \mathcal{L}(E, F)$. Then the following are equivalent:

- (i) $|\Phi f| \leq \Psi|f|$ for all $f \in E$, i.e. Ψ dominates Φ , and
- (ii) $|T_\Phi(t)f| \leq T_\Psi(t)|f|$ for all $f \in E, t \geq 0$, i.e. $(T_\Psi(t))_{t \geq 0}$ dominates $(T_\Phi(t))_{t \geq 0}$.

Proof. “(i) \Rightarrow (ii)” We show that the resolvent of A_Φ is dominated by the resolvent of A_Ψ (see [32, Proposition C-II, 4.1]). For large λ we have

$$R(\lambda, A_\Phi)f = \varepsilon_\lambda \otimes R(1, \Phi_\lambda)\Phi R(\lambda, A_0)f + R(\lambda, A_0)f$$

and

$$R(\lambda, A_\Psi)f = \varepsilon_\lambda \otimes R(1, \Psi_\lambda)\Psi R(\lambda, A_0)f + R(\lambda, A_0)f.$$

Let $f \in E$. Then the following estimate holds:

$$\begin{aligned} |R(\lambda, A_\Phi)f| &\leq |\varepsilon_\lambda \otimes R(1, \Phi_\lambda)\Phi R(\lambda, A_0)f| + |R(\lambda, A_0)f| \\ &\leq \varepsilon_\lambda \otimes |R(1, \Phi_\lambda)\Phi R(\lambda, A_0)f| + R(\lambda, A_0)|f| \\ &\leq \varepsilon_\lambda \otimes R(1, \Psi_\lambda)|\Phi R(\lambda, A_0)f| + R(\lambda, A_0)|f| \\ &\leq \varepsilon_\lambda \otimes R(1, \Psi_\lambda)\Psi R(\lambda, A_0)|f| + R(\lambda, A_0)|f| \\ &= R(\lambda, A_\Psi)|f|. \end{aligned}$$

Thus $R(\lambda, A_\Phi)$ is dominated by $R(\lambda, A_\Psi)$.

“(ii) \Rightarrow (i)” The assertion is a consequence of the translation property, thus recall that

$$(7) \quad T_\Phi(t)g(x) = \begin{cases} g(x+t) & \text{if } x+t \leq 0, \\ \Phi(T_\Phi(x+t)g) & \text{if } x+t > 0, \end{cases} \quad \text{for all } g \in E.$$

Let $f \in E$. By (ii) we have $|T_\Phi(t)f| \leq T_\Psi(t)|f|$ for all $t > 0, f \in E$. Hence $|T_\Phi(t)f(x)| \leq T_\Psi(t)|f|(x)$ for almost every $x \in (-\infty, 0]$. By formula (7) we get

$$|\Phi(T_\Phi(t+x)f)| \leq \Psi(T_\Psi(t+x)|f|) \quad \text{for } x \text{ with } x+t > 0.$$

Let $\varepsilon > 0$ and let t, x be given with $t+x = \varepsilon$ such that $|\Phi(T_\Phi(\varepsilon)f)| \leq \Psi(T_\Psi(\varepsilon)|f|)$ for x with $x+t > 0$. By the continuity of $|\cdot|$ and Φ we have that $\lim_{\varepsilon \rightarrow 0} |\Phi(T_\Phi(\varepsilon)f)| = |\Phi(f)|$ and $\lim_{\varepsilon \rightarrow 0} |\Psi(T_\Psi(\varepsilon)f)| = |\Psi(f)|$. Thus the above estimate implies $|\Phi(f)| \leq \Psi|f|$. \square

The following result is proved in [3].

Lemma 6. Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on a Banach lattice E with order continuous norm. If there exists a dominating semigroup, then there exists a modulus semigroup of $(T(t))_{t \geq 0}$.

Proposition 3.10. Let F be a Banach lattice with order continuous norm and $E = L^1((-\infty, 0], F; e^{\eta s} ds)$. Let $\Phi \in \mathcal{L}(E, F)$ be a regular operator (i.e.

$\Phi = \Phi_1 - \Phi_2$ where Φ_i is positive ($i = 1, 2$). Then the semigroup $(T_{|\Phi|}(t))_{t \geq 0}$ is the modulus semigroup of $(T_\Phi(t))_{t \geq 0}$.

Proof. Since Φ is regular we have that $|\Phi|$, given by

$$|\Phi|h = \sup\{|\Phi g| : |g| \leq h\}$$

for $h \in E_+$, exists. Moreover $|\Phi|$ is the smallest bounded linear operator majorizing Φ (see [35, Chapter IV.1]). The first assertion that $(T_{|\Phi|}(t))_{t \geq 0}$ dominates $(T_\Phi(t))_{t \geq 0}$ thus follows from Lemma 5, since $|\Phi f| \leq |\Phi||f|$, thus $|T_\Phi(t)f| \leq T_{|\Phi|}(t)|f|$ for all $f \in E$. Moreover we also showed in Lemma 5 (“(ii) \Rightarrow (i)”) that $(T_{|\Phi|}(t))_{t \geq 0}$ is the smallest semigroup of the form $(T_\Psi(t))_{t \geq 0}$ which dominates $(T_\Phi(t))_{t \geq 0}$.

The assumption that \bar{F} is a Banach lattice with order continuous norm implies that E is a Banach lattice with order continuous norm as well. By Lemma 6 there exists a modulus semigroup $(S(t))_{t \geq 0}$ of $(T_\Phi(t))_{t \geq 0}$. Thus we have

$$|T_\Phi(t)f| \leq S(t)|f| \leq T_{|\Phi|}(t)|f| \quad \text{for all } f \in E,$$

hence

$$|T_\Phi(t)f(x)| \leq S(t)|f|(x) \leq T_{|\Phi|}(t)|f|(x) \quad \text{for almost every } x \in (-\infty, 0].$$

By the translation property (4), which is valid for $(T_\Phi(t))_{t \geq 0}$ and for $(T_{|\Phi|}(t))_{t \geq 0}$ we thus have

- (a) $|f(x+t)| \leq S(t)|f|(x) \leq |f|(x+t)$ if $x \leq -t$; and
- (b) $|\Phi(T_\Phi(t+x)f)| \leq S(t)|f|(x) \leq |\Phi|(T_{|\Phi|}(t+x)f)$ if $x \geq -t$.

By estimate (a) we get $S(t)|f|(x) = |f(x+t)|$ for $x \leq -t$. Since $f = f^+ - f^-$, we have $S(t)f(x) = S(t)f^+(x) - S(t)f^-(x) = f^+(x+t) - f^-(x+t) = f(x+t)$ if $x \leq -t$. Thus $(S(t))_{t \geq 0}$ is a translation semigroup on E (see Definition 1.3).

Let $t \geq 0$ fixed. For almost every $x \geq -t$ we have by estimate (b) that

$$\begin{aligned} (8) \quad \|S(t)f(x)\|_F &= \|S(t)f^+(x) - S(t)f^-(x)\|_F \\ &\leq \|S(t)f^+(x)\|_F + \|S(t)f^-(x)\|_F \\ &\leq \|\Phi\| \cdot M_1 \cdot \|f^+\|_E + \|\Phi\| \cdot M_2 \cdot \|f^-\|_E \\ &\leq M_3 \cdot (\|f^+\|_E + \|f^-\|_E) \leq M\|f\|_E \end{aligned}$$

for constants M_1, M_2, M_3 and M and all $f \in E$.

Let B be the generator corresponding to the semigroup $(S(t))_{t \geq 0}$. Then by Proposition 1.4 $D(B) \subseteq W^{1,1}$ and $Bf = f'$ for $f \in D(B)$. Since $f \in D(B)$ implies $S(t)f \in D(B)$, we have that $S(t)f$ is continuous.

We thus can evaluate $S(t)f$ at 0 and moreover obtain by the above estimate (8) that $\|S(t)f(0)\|_F \leq M\|f\|_E$ and especially $\|f(0)\|_F \leq M\|f\|_E$. Thus $f \mapsto f(0)$ is a continuous map from $(D(B), \|\cdot\|_E)$ into F , thus by continuation $f(0) = \Psi(f)$ for an operator $\Psi \in \mathcal{L}(E, F)$ whenever $f \in D(B)$. Hence $D(B) \subseteq \{f \in W^{1,1}: f(0) = \Psi(f)\} =: D$. On the other hand we know by Theorem 2.3 that $Bf = f'$ with domain D is a generator, thus $D(B) = D$ and $B = A_\Psi$. As already remarked this implies $\Psi = |\Phi|$, since $(T_{|\Phi|}(t))_{t \geq 0}$ is the smallest semigroup of this form which dominates $(T_\Phi(t))_{t \geq 0}$. \square

Proposition 3.11. *Let Φ be a Lipschitz-continuous operator from E into F and let $\Psi \in \mathcal{L}(E, F)$ be linear such that $|\Phi f| \leq \Psi|f|$ for all $f \in E$. Then $|T_\Phi(t)f| \leq T_\Psi(t)|f|$ for all $f \in E$.*

One can prove this result similarly to “(i) \Rightarrow (ii)” of Lemma 5 using the operators J_α^Φ instead of the resolvent operator $R(\lambda, A_\Phi)$. By estimating $|J_\alpha^\Phi f| - J_\alpha^\Psi|f|$ one obtains using Gronwall's inequality that

$$(\text{Id} - \varepsilon_{1/\alpha} \otimes \Psi)(|J_\alpha^\Phi f| - J_\alpha^\Psi|f|) \leq 0.$$

Then $|J_\alpha^\Phi f| - J_\alpha^\Psi|f| \leq 0$ follows, since for small α the operator $(\text{Id} - \varepsilon_{1/\alpha} \otimes \Psi)^{-1}$ exists and is positive. (For details see [18].)

(e) Localization of the spectrum. Let F be a Banach lattice. Furthermore let $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ as usual. By Proposition 3.4 we know that the positivity of Φ is equivalent to the positivity of the semigroup $(T_\Phi(t))_{t \geq 0}$. Thus, using the Perron-Frobenius theory for positive operator semigroups and especially the results of G. Greiner (see [32, Chapters C–III and C–IV]) one can prove the following proposition which characterizes the spectrum of the generator A_Φ on the Banach lattice E in terms of the operators Φ_λ on the much “smaller” Banach lattice F . Indeed, if $F = \mathbb{R}$ the location of the spectral bound $s(A_\Phi)$ reduces to solving a simple real valued equation, also called the *characteristic equation*, since it most often characterizes the asymptotic behavior of the semigroup.

For a moment let us consider the general setting, where F is a Banach space and $\Phi \in \mathcal{L}(E, F)$ is not necessarily positive.

It is easy to determine the point spectrum of A_Φ . One can show that for $\lambda > -\eta$

$$\lambda \in P\sigma(A_\Phi) \quad \text{if and only if} \quad 1 \in P\sigma(\Phi_\lambda).$$

(In all considerations concerning the spectrum $-\eta$ is a distinguished point, since $\sigma(A_0) = \{\lambda \in \mathbb{R} : \text{Re } \lambda \leq -\eta\}$ as one easily verifies.)

Indeed $A_\Phi f = \lambda f$ with $D(A_\Phi) \ni f \neq 0$ implies that $\Phi_\lambda f(0) = f(0)$ and conversely $\Phi_\lambda x = x$ with $F \ni x \neq 0$ implies that $A_\Phi f = \lambda f$ for $f := \varepsilon_\lambda \otimes x$.

A similar relation holds for the entire spectrum and for the spectral bound of A_Φ if F is a Banach lattice and $\Phi \geq 0$. Summarizing we can state the following result:

Proposition 3.12. *Let F be a Banach space, $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ and $\Phi \in \mathcal{L}(E, F)$. Then $1 \in \varrho(\Phi_\lambda)$ implies $\lambda \in \varrho(A_\Phi)$.*

Under the additional assumption that $\Phi(D(A_0)) = F$ or that Φ_λ is compact for all $\lambda > -\eta$ we have that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > -\eta$ the following equivalence holds:

$$(9) \quad \lambda \in \sigma(A_\Phi) \quad \text{if and only if} \quad 1 \in \sigma(\Phi_\lambda).$$

If F is a Banach lattice and Φ a positive operator and additionally $\lambda \mapsto r(\Phi_\lambda)$ is a strictly decreasing, continuous map ($r(\Phi_\lambda)$ spectral radius of Φ_λ), then the spectral bound of A_Φ can be determined by the equivalence:

$$(10) \quad \begin{array}{ccc} < & & < \\ \lambda & = & s(A_\Phi) & \text{if and only if} & 1 & = & r(\Phi_\lambda). \\ > & & & & & & > \end{array}$$

In particular $\lambda_0 = s(A_\Phi)$ is the unique real solution of $r(\Phi_\lambda) = 1$.

We will refer to (10) sometimes as the “characteristic equation”.

The proof of this proposition for the case $E = L^1([-1, 0], F)$ can be found in [32, Proposition C–IV, 3.6] and can easily be modified to fit our situation, where $E = L^1((-\infty, 0], F; e^{\eta s} ds)$. In more detail the case $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ is treated in [18].

It should be remarked that for positive Φ the map $\lambda \mapsto s(\Phi_\lambda)$ is left-continuous and nonincreasing which follows from the positivity assumption on Φ . Compactness of the operators Φ_λ implies continuity of the map $\lambda \mapsto r(\Phi_\lambda)$ (see [12]).

(f) Linearization in the nonlinear case. One method used to study stability properties of equilibrium solutions of nonlinear differential equations is to consider the linearization with respect to this equilibrium. This concept has analogues in the theory of strongly continuous semigroups. One method is developed by W. Desch and W. Schappacher in [9] and we want to apply this in our situation.

Further results concerning this concept of “linearized stability” are developed by G. Greiner [20] and Ph. Clément et al. [6]. We need the following definition:

Definition 3.13. (a) *Let $(T(t))_{t \geq 0}$ be a nonlinear semigroup on a Banach space E . An element $\bar{x} \in E$ is called an equilibrium of $(T(t))_{t \geq 0}$ if $T(t)\bar{x} = \bar{x}$ for all $t \geq 0$.*

(b) *An equilibrium $\bar{x} \in E$ is called (Liapunov)-stable if for any neighborhood U of \bar{x} there exist a neighborhood V of \bar{x} such that $T(t)x \in U$ for any $x \in V$ and all $t \geq 0$. If additionally there exists $\delta > 0$ and $M \geq 0$ such that $\|T(t)x - \bar{x}\| \leq Me^{-\delta t}$, then the equilibrium \bar{x} is called an exponentially stable equilibrium of $(T(t))_{t \geq 0}$.*

The result of W. Desch and W. Schappacher which we will use is the following:

Proposition 3.14. *Let $(T(t))_{t \geq 0}$ be a nonlinear semigroup on a Banach space E and let \bar{x} be an equilibrium. Suppose that $(T(t))_{t \geq 0}$ is Fréchet-differentiable at \bar{x} with $S(t) = T'(t, \bar{x})$ (Fréchet-derivative of $T(t)$ at \bar{x}). (Then $(S(t))_{t \geq 0}$ is a linear semigroup.) If the zero solution of the semigroup $(S(t))_{t \geq 0}$ is exponentially stable (i.e. $\omega(S(t)) < 0$) then \bar{x} is exponentially stable with respect to $(T(t))_{t \geq 0}$.*

To use these results in our case of determining asymptotic properties of the nonlinear semigroup $(T_\Phi(t))_{t \geq 0}$ we have to find the linearization of $(T_\Phi(t))_{t \geq 0}$ in an equilibrium point \bar{f} first. We suppose that 0 is an equilibrium solution and later on reduce the case of a nonzero equilibrium to this case.

Proposition 3.15. *Let F be a Banach space and let*

$$E = L^1((-\infty, 0], F; e^{\eta s} ds)$$

as usual. Furthermore let Φ be a Lipschitz-continuous map from E into F such that $\Phi(0) = 0$ and Φ is Fréchet-differentiable in 0 . Then 0 is an equilibrium of $T_\Phi(t)$ and the linearized semigroup of $(T_\Phi(t))_{t \geq 0}$ is the semigroup $(T_{\Phi'(0)}(t))_{t \geq 0}$ where $\Phi'(0) \in \mathcal{L}(E, F)$ denotes the Fréchet-derivative of Φ in 0 . As usual $(T_{\Phi'(0)}(t))_{t \geq 0}$ is the semigroup with generator $A_{\Phi'(0)}f = f'$ and domain $D(A_{\Phi'(0)}) = \{f \in W^{1,1} : f(0) = [\Phi'(0)](f)\}$.

Proof. Since $\Phi(0) = 0$ we have $T_\Phi(t)0 = 0$ by the translation property. To show that $(T_{\Phi'(0)}(t))_{t \geq 0}$ is the linearization of $(T_\Phi(t))_{t \geq 0}$ in 0 we have to verify that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $g \in E$ with $\|g\| < \delta$ we have

$$\|T_\Phi(t)g - T_{\Phi'(0)}(t)g\|_E < \varepsilon \|g\|_E.$$

Using formula (4) for the linear semigroup $(T_{\Phi'(0)}(t))_{t \geq 0}$ and the nonlinear semigroup $(T_\Phi(t))_{t \geq 0}$ respectively, we obtain the following estimate.

$$\begin{aligned} \|T_\Phi(t)g - T_{\Phi'(0)}(t)g\|_E &= \int_{-\infty}^0 \|T_\Phi(t)g(x) - T_{\Phi'(0)}(t)g(x)\|_F dx \\ &= \int_{-\infty}^{-t} \|g(x+t) - g(x+t)\|_F dx \\ &\quad + \int_{-t}^0 \|\Phi(T_\Phi(t+x)g) - \Phi'(0)[T_{\Phi'(0)}(t+x)g]\|_F dx \\ &= \int_{-t}^0 \|\Phi(T_\Phi(t+x)g) - \Phi'(0)[T_{\Phi'(0)}(t+x)g]\|_F dx \end{aligned}$$

$$\begin{aligned}
&\leq \int_{-t}^0 \left\| \Phi(T_\Phi(t+x)g) - \Phi'(0)[T_\Phi(t+x)g] \right\|_F dx \\
&\quad + \left\| \Phi'(0) \right\| \cdot \int_{-t}^0 \left\| T_\Phi(t+x)g - T_{\Phi'(0)}(t+x)g \right\|_E dx \\
&\leq \int_0^t \left\| \Phi(T_\Phi(s)g) - \Phi'(0)[T_\Phi(s)g] \right\|_F ds \\
&\quad + \left\| \Phi'(0) \right\| \cdot \int_0^t \left\| T_\Phi(s)g - T_{\Phi'(0)}(s)g \right\|_E ds.
\end{aligned}$$

Let $\varepsilon > 0$. Since Φ is Fréchet-differentiable in 0 with Fréchet-derivative $\Phi'(0)$, there exists a $\delta_1 > 0$ such that $\|T_\Phi(s)g\|_E < \delta_1$ implies that

$$\left\| \Phi(T_\Phi(s)g) - \Phi'(0)[T_\Phi(s)g] \right\|_F \leq \varepsilon \|T_\Phi(s)g\|_E \quad \text{for all } 0 \leq s \leq t.$$

By the continuity of the map $g \mapsto T_\Phi(s)g$ there exists δ_0 such that $\|g\|_E < \delta_0$ implies that $\|T_\Phi(s)g\|_E = \|T_\Phi(s)g - T_\Phi(s)0\|_E < \delta_1$ for all $0 \leq s \leq t$. Then $\|g\|_E < \delta_0$ implies that $\|T_\Phi(s)g\|_E < \delta_1$, but this implies that $\left\| \Phi(T_\Phi(s)g) - \Phi'(0)[T_\Phi(s)g] \right\|_F \leq \varepsilon \|T_\Phi(s)g\|_E$ for all $0 \leq s \leq t$ as we noted above.

Thus we get by the above estimate that

$$\begin{aligned}
&\left\| T_\Phi(t)g - T_{\Phi'(0)}(t)g \right\|_E \\
&\leq M \cdot \int_0^t \left\| T_\Phi(s)g - T_{\Phi'(0)}(s)g \right\|_E ds + \int_0^t \varepsilon \cdot \|T_\Phi(s)g\|_E ds \\
&\leq M \cdot \int_0^t \left\| T_\Phi(s)g - T_{\Phi'(0)}(s)g \right\|_E ds + \varepsilon \cdot t \cdot e^{\|\Phi\|_{\text{Lip}} t} \|g\| \\
&\leq M \cdot \int_0^t \left\| T_\Phi(s)g - T_{\Phi'(0)}(s)g \right\|_E ds + \alpha(t),
\end{aligned}$$

where $\alpha(t) = \varepsilon \cdot t \cdot e^{\|\Phi\|_{\text{Lip}} t} \|g\|$ and $M = \left\| \Phi'(0) \right\|$.

By the lemma of Gronwall we thus have

$$\begin{aligned}
&\left\| T_\Phi(t)g - T_{\Phi'(0)}(t)g \right\|_E \leq \alpha(t) + M \cdot \int_0^t e^{M(t-s)} \alpha(s) ds \\
&= \varepsilon \cdot t \cdot e^{\|\Phi\|_{\text{Lip}} t} \|g\| + M \cdot \int_0^t e^{M(t-s)} \varepsilon \cdot s \cdot e^{\|\Phi\|_{\text{Lip}} s} \|g\| ds \\
&\leq \varepsilon(C_1 + C_2) \|g\|
\end{aligned}$$

for constants C_1, C_2 (depending on t).

This shows that for all g with $\|g\| < \delta_0$ we have

$$\left\| T_\Phi(t)g - T_{\Phi'(0)}(t)g \right\|_E \leq \varepsilon(C_1 + C_2) \|g\|.$$

In other words $T_{\Phi'(0)}(t)$ is the Fréchet-derivative of $T_{\Phi}(t)$ in 0. By our general considerations in §2 we know that $(T_{\Phi'(0)}(t))_{t \geq 0}$ is a strongly continuous semigroup. Thus we have that $(T_{\Phi'(0)}(t))_{t \geq 0}$ is the linearized semigroup of $(T_{\Phi}(t))_{t \geq 0}$ in the equilibrium 0. \square

4. ASYMPTOTIC BEHAVIOR OF $(T_{\Phi}(t))_{t \geq 0}$ AND FINAL REMARKS

In the previous section we studied diverse properties of $(T_{\Phi}(t))_{t \geq 0}$ under conditions posed on the operator Φ . Here we want to state a convergence result which is the direct consequence of these properties as it is discussed in more detail elsewhere (see for example [32, Sections B-IV, 2 and C-IV, 2]). Eventually we will mention some examples where these results can be applied.

First we consider the case where $\Phi \in \mathcal{L}(E, F)$ is a linear positive operator from E into F . For nonpositive operators one can apply these results to a majorizing semigroup, resp. to the modulus semigroup, instead. Secondly we state a result for the nonlinear case which uses Proposition 3.14 and Proposition 3.15.

Our main result in the linear case is the following.

Theorem 4.1. *Let F be a Banach lattice, $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ and $\Phi \in \mathcal{L}(E, F)$ a positive linear operator. Furthermore assume that $\Phi \circ A_0$ has a continuous extension from E into F . Suppose Φ is compact, strictly positive and that the operators $\Phi_{\lambda} \in \mathcal{L}(F)$ given by $\Phi_{\lambda}x = \Phi(e^{\lambda \cdot} x)$ are irreducible for large λ . Suppose $s(A_{\Phi}) > -\eta$. Furthermore assume that $\lambda \mapsto r(\Phi_{\lambda})$ is a strictly decreasing map from $(-\eta, \infty)$ to \mathbb{R} . Then the following assertions hold:*

- (1) *There exists a unique real number $\lambda_0 > -\eta$ satisfying $r(\Phi_{\lambda_0}) = 1$. This number λ_0 coincides with the spectral bound $s(A_{\Phi}) = \omega(A_{\Phi})$.*
- (2) *There exists a strictly positive projection P of rank 1 such that $\|e^{-\lambda_0 t} T_{\Phi}(t) - P\| \leq M e^{-\delta t}$ for suitable constants $\delta > 0, M \geq 1$ and all $t \geq 0$.*

Remarks. (a) It is possible to weaken the strong positivity assumption on Φ . Indeed the same assertions hold under the assumptions of Proposition 3.8 which assure the irreducibility of $(T_{\Phi}(t))_{t \geq 0}$.

(b) Let Φ be a weakly compact operator and F be a Banach lattice such that $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ is a Banach lattice with the Dunford-Pettis property (thus especially for $F = L^1(X, \Sigma, \mu)$, see also [35, Chapter II.9]). Under these assumptions the conclusions of Theorem 4.1 remain true. This can be verified using the fact that in Banach spaces with the Dunford-Pettis property the power T^2 of a weakly compact operator T is compact in the strong topology as well. (For more details see [35 and 1].)

(c) The assertion (2) of Theorem 4.1 can be formulated in the following way: There exists a $0 \neq c \in [L^1((-\infty, 0], F; e^{\eta s} ds)]'$ and a quasi-interior element g

of $L^1((-\infty, 0], F; e^{\eta s} ds)_+$ such that

$$(11) \quad T_\Phi(t)f = c(f)e^{\lambda_0 t}g + o(e^{\lambda_0 t}) \quad \text{for } t \geq 0 \text{ and each } f \in E.$$

For the corresponding Cauchy problem this means that there exists an *exponentially steady state* with *intrinsic growth constant* λ_0 (see for example [31 and 43] for this notation).

(d) The standard way to use Theorem 4.1 is to show that there exists $\lambda_0 > 0$ such that $r(\Phi_{\lambda_0}) = 1$ in order to conclude that $s(A_\Phi) = \lambda_0$. This can very often be accomplished by showing that $r(\Phi_0) > 1$ and thus the monotonicity of $\lambda \mapsto r(\Phi_\lambda)$ shows the existence of $\lambda_0 > 0$ satisfying $r(\Phi_{\lambda_0}) = 1$. In contrast to “nonpositive” semigroups, here one only has to consider real λ .

In the case where Φ is nonlinear but Lipschitz-continuous we can formulate the following result.

Proposition 4.2. *Let $E = L^1((-\infty, 0], F; e^{\eta s} ds)$, where F is a Banach lattice. Let Φ be a Lipschitz-continuous map from E into F . Suppose that there exists a linear operator $\Psi \in \mathcal{L}(E, F)$ such that $|\Phi f| \leq \Psi|f|$ for all $f \in E$. Furthermore let Ψ satisfy the assumptions of Theorem 4.1. Then*

$$(12) \quad \|T_\Phi(t)f\| \leq M \cdot e^{\lambda_0 t} \|f\| + o(e^{\lambda_0 t}).$$

Proof. By Proposition 3.10 we have that $|T_\Phi(t)f| \leq T_\Psi(t)|f|$ for all $f \in E, t \geq 0$. Moreover we know that there exists $g \in E_+, c \in (L^1((-\infty, 0], F; e^{\eta s} ds))'$ such that $T_\Psi(t)f = c(f)e^{\lambda_0 t}g + o(e^{\lambda_0 t})$ for $t \geq 0$ and each $f \in E$, thus

$$\begin{aligned} \lim_{t \rightarrow \infty} \|e^{-\lambda_0 t} T_\Phi(t)f\| &\leq \lim_{t \rightarrow \infty} \|e^{-\lambda_0 t} T_\Psi(t)|f|\| \\ &\leq \|c(|f|)g\| \leq M \cdot \|f\| \end{aligned}$$

or equivalently that

$$\|T_\Phi(t)f\| \leq e^{\lambda_0 t} \cdot M \cdot \|f\| + o(e^{\lambda_0 t}). \quad \square$$

More detailed information can be obtained by using the “linearized stability” result of Proposition 3.14.

Proposition 4.3. *Let F be a Banach lattice, $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ and let Φ be a Lipschitz-continuous map from E to F such that $\Phi(0) = 0$ and Φ is Fréchet-differentiable in 0 . Let $\Psi := \Phi'(0) \in \mathcal{L}(E, F)$ denote the Fréchet-derivative of Φ in 0 . Suppose the semigroup $(T_\Psi(t))_{t \geq 0}$ is uniformly exponentially stable, that is $\omega(T_\Psi(t)) < 0$. Then the zero-equilibrium of the nonlinear semigroup $(T_\Phi(t))_{t \geq 0}$ is exponentially stable.*

Proof. The result follows directly from Propositions 3.14 and 3.15. \square

Remark. 1. The assertions on Φ , that is the Lipschitz-continuity and the Fréchet-differentiability in 0 , are satisfied, for example, under the assumption that Φ is everywhere Fréchet-differentiable and Φ' is uniformly bounded.

2. Let $\Psi = \Phi'(0)$, then $r(\Psi_0) < 1$ implies that $s(A_\Psi) < 0$, since $\lambda \mapsto r(\Psi_\lambda)$ is a decreasing function. Thus, if Ψ is positive and if $\omega(A_\Psi) = s(A_\Psi)$ which is true for many positive semigroups (see [32]), then $r(\Psi_0) < 1$ is sufficient to show exponential stability of the zero-equilibrium.

To allow nonzero-equilibria, we consider the space $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ where $\eta > 0$. Let $\mathbb{1}$ denote the function in $L^1((-\infty, 0], \mathbb{R}, e^{\eta s} ds)$ which is the constant 1, thus $\mathbb{1}(s) = 1$ for all $s \in (-\infty, 0]$. Observing the fact that nonzero-equilibria of the nonlinear semigroup $(T_\Phi(t))_{t \geq 0}$ are generally of the form $\bar{f} = \mathbb{1} \otimes \bar{x}$, where \bar{x} is an equilibrium of the operator Φ_0 on F , that is $\Phi_0 \bar{x} = \Phi(\mathbb{1} \otimes \bar{x}) = \bar{x}$, one can deduce this case from the above situation.

Indeed let $\bar{f} = \mathbb{1} \otimes \bar{x}$ be an equilibrium of $(T_\Phi(t))_{t \geq 0}$, thus $T_\Phi(t)\bar{f} = \bar{f}$. Define $S(t)$ by $S(t)h = T_\Phi(t)(h + \bar{f}) - \bar{f}$ for $h \in E$. Clearly $(S(t))_{t \geq 0}$ is a strongly continuous semigroup as well and $S(t)0 = 0$. First we will show that $S(t) = T_{\tilde{\Phi}}(t)$ where $\tilde{\Phi}$ is a map from E into F defined by $\tilde{\Phi}h = \Phi(h + \bar{f}) - \bar{x}$. Secondly we use Proposition 4.3 to show local stability of the equilibrium \bar{f} . We have to show that $(S(t))_{t \geq 0}$ satisfies the translation property (4) with respect to the operator $\tilde{\Phi}$. The following holds:

$$\begin{aligned} S(t)h(s) &= T_\Phi(t)(h + \bar{f})(s) - \bar{f}(s) \\ &= \begin{cases} (h + \bar{f})(t+s) - \bar{f}(s) & \text{if } t+s < 0, \\ \Phi(T_\Phi(t+s)(h + \bar{f})) - \bar{x} & \text{if } t+s \geq 0, \end{cases} \\ &= \begin{cases} h(t+s) + \bar{f}(t+s) - \bar{f}(s) & \text{if } t+s < 0, \\ \Phi(S(t+s)h + \bar{f}) - \bar{x} & \text{if } t+s \geq 0, \end{cases} \\ &= \begin{cases} h(t+s) & \text{if } t+s < 0 \\ \tilde{\Phi}(S(t+s)h) & \text{if } t+s \geq 0 \end{cases} \quad \text{by the definition of } \tilde{\Phi}. \end{aligned}$$

Thus $S(t) = T_{\tilde{\Phi}}(t)$ and hence

$$T_\Phi(t)h = T_{\tilde{\Phi}}(t)(h - \bar{f}) + \bar{f}.$$

It is clear that $\tilde{\Phi}$ is Lipschitz-continuous, resp. Fréchet-differentiable, if Φ has this property. Furthermore $\tilde{\Phi}0 = \Phi(\bar{f}) - \bar{x} = 0$, thus 0 is an equilibrium of $\tilde{\Phi}$. We can apply Proposition 4.3 to the semigroup $(T_{\tilde{\Phi}}(t))_{t \geq 0}$ and obtain a local stability result for $(T_{\tilde{\Phi}}(t))_{t \geq 0}$ under the appropriate conditions on $\tilde{\Phi}$, resp. Φ . Since the Fréchet-derivative of Φ in $\mathbb{1} \otimes \bar{x}$ is identical to the Fréchet-derivative of $\tilde{\Phi}$ in 0 we altogether obtain the following result.

Theorem 4.4. *Let F be a Banach lattice, $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ and Φ be a Lipschitz-continuous map from E to F . Let $\bar{x} \in F$ such that $\Phi(\mathbb{1} \otimes \bar{x}) = \bar{x}$ and let $\Psi := \Phi'(\mathbb{1} \otimes \bar{x}) \in \mathcal{L}(E, F)$ denote the Fréchet-derivative of Φ in $\mathbb{1} \otimes \bar{x}$. Suppose the semigroup $(T_\Psi(t))_{t \geq 0}$ is uniformly exponentially stable, that is $\omega(T_\Psi(t)) < 0$, then the equilibrium $\mathbb{1} \otimes \bar{x}$ of the nonlinear semigroup $(T_\Phi(t))_{t \geq 0}$ is exponentially stable.*

After having developed this general situation we will give some examples—one in some detail—where these results can be applied. There are at least three classes of equations which fit directly into this setting.

- (a) Population equations,
- (b) Renewal equations,
- (c) Volterra equations.

(a) Population equations. The starting point of many population models is an age-dependent nonlinear population equation whose formulation goes back to Sharpe-Lotka [36] and reads as follows:

$$\begin{aligned}
 (13) \quad & \frac{\partial}{\partial t} n(t, a) + \frac{\partial}{\partial a} n(t, a) = -\mu(a, P(t)) n(t, a), \\
 & n(t, 0) = \int_0^\infty \beta(a, P(t)) n(t, a) da, \\
 & n(0, a) = n_0(a), \\
 & P(t) = \int_0^\infty n(t, a) da.
 \end{aligned}$$

If we neglect the death rate μ , that is, if we consider (13) for $\mu = 0$, then we end up with the Cauchy problem corresponding to the translation semigroup $(T_\Phi(t))_{t \geq 0}$ with generator A_Φ , where Φ is \mathbb{R}^n -valued and given by

$$\Phi(f) = \int_0^\infty \beta(a, \int_0^\infty f(a) da) f(a) da.$$

(Here $F = \mathbb{R}^n$ and $E = L^1(\mathbb{R}_+, \mathbb{R}^n)$. Notice that here we consider functions on \mathbb{R}_+ instead of \mathbb{R}_- .)

If we include a nonzero death rate μ in (13), then the solutions of this Cauchy problem again form a nonlinear semigroup $(T_{\Phi, \mu}(t))_{t \geq 0}$ on the space of initial functions. Moreover the “translation property” (formula (4) from §2) now reads as

$$T_{\Phi, \mu}(t)f(a) = \begin{cases} f(a-t)e^{-\int_{a-t}^a \mu(r, P(t)) dr} & \text{if } a-t \geq 0, \\ \Phi(T_{\Phi, \mu}(t-a)f)e^{-\int_0^a \mu(r, P(t)) dr} & \text{if } a-t \leq 0. \end{cases}$$

This example is intensely studied, e.g. in [34 and 42], and thus we are not giving any details here. Instead, we will give a list of possible operators Φ from $E = L^1(\mathbb{R}_+, \mathbb{R}^n)$ to \mathbb{R}^n arising in population dynamics (see [42] for details).

- (a) $\Phi(f) = \int_0^\infty \beta(Pf)\gamma(a)f(a) da,$
- (b) $\Phi(f) = \int_0^\infty \beta(a, Pf)f(a) da,$
- (c) $\Phi(f) = \int_0^\infty \beta(a)g(a, f(a)) da,$

where $Pf = \int_0^\infty f(s) ds$. Here the corresponding "linearizations" (again near equilibrium solutions of the form $\bar{f} = \mathbb{1} \otimes \bar{x}$ satisfying $\Phi(\mathbb{1} \otimes \bar{x}) = \bar{x}$) are given by

$$\begin{aligned} (a') \quad & \Phi'(\bar{f})(f) = \beta'(P\bar{f})Pf \int_0^\infty \gamma(a)\bar{f}(a) da + \beta(P\bar{f}) \int_0^\infty \gamma(a)f(a) da. \\ (b') \quad & \Phi'(\bar{f})(f) = \int_0^\infty \partial_2 \beta(a, P\bar{f})\bar{f}(a) da \cdot (Pf) + \int_0^\infty \beta(a, P\bar{f})f(a) da. \\ (c') \quad & \Phi'(\bar{f})(f) = \int_0^\infty \beta(a)\partial_2 g(a, \bar{f}(a))f(a) da. \end{aligned}$$

(b) Renewal equations. As already remarked in the introduction first derivatives and equivalently translation semigroups are closely related to renewal, resp., delay equations.

Let F be a Banach space and let $E = L^1((-\infty, 0], F; e^{\eta s} ds)$. For a function $u \in X$ where $X = \bigcap_{a < \infty} L^1((-\infty, a], F; e^{\eta s} ds)$ we denote by u_t the function $u_t \in E = L^1((-\infty, 0], F; e^{\eta s} ds)$ for which $u_t(s) = u(t+s)$ for all $s \leq 0$. Thus u_t describes the "history" of the state function u at time t and is considered as a function on \mathbb{R}_- .

We call a *renewal equation* a functional equation of the form

$$(14) \quad u(t) = \Phi(u_t), \quad u_0 = \varphi,$$

where Φ is a Lipschitz continuous operator from E to F . Thus u is a function with values in F which is Bochner integrable on every interval $[0, a]$. The connection between (14) and the translation semigroup $(T_\Phi(t))_{t \geq 0}$ studied in §§2 and 3 is given by the formula

$$(15) \quad T_\Phi(t)\varphi = u_t.$$

This connection uniquely defines the semigroup $(T_\Phi(t))_{t \geq 0}$ for a given solution of the renewal equation (14) and vice versa. Using this correspondence the asymptotic behavior of solutions of (14) can be determined by studying the asymptotic behavior of the semigroup $(T_\Phi(t))_{t \geq 0}$. Such questions have been discussed in [45 and 44]. The class of equations of the form (14) includes very different examples. As studied by G. F. Webb [44] cell population equations with age and size structure can very often be reformulated to yield such a renewal equation. To do so one has to consider operators Φ from E into F (where F is a function space on \mathbb{R}_+) given by

$$(16) \quad \Phi(\varphi)(x) = 2 \int_0^\infty \int_0^\infty k(s(a, x), u) \frac{g(s(a, x))}{g(x)} \cdot f(a, s(a, x)) \mu(a, s(a, x)) \varphi(-a)(u) du da,$$

or

$$(17) \quad \Phi(\varphi)(x) = 4 \int_0^\infty \frac{g(s(a, x))}{g(x)} f(a, s(a, x)) \mu(a, s(a, x)) \varphi(-a)(2s(a, x)) da.$$

(See [44] for the choice of an appropriate state space F , resp., E and for the choice of the parameter functions k, g, f, μ and s .)

A related example is a cell cycle model which is formulated and investigated by O. Arino and M. Kimmel in [28 and 2]. Their model leads to an equation of the form (14), where Φ is given by

$$(18) \quad \Phi(\varphi)(x) = \int_0^\infty k(x, a)\varphi(q(x))(a) da.$$

Here an appropriate choice for the Banach space F is $F = L^1(\mathbb{R}_+)$. This model fits into our situation and the mathematical discussion simplifies considerably (see also [18]). A detailed analysis of this model using the results from §§2 and 3 will be done separately elsewhere.

(c) Volterra equations. We want to discuss in a little more detail how Volterra equations (linear and nonlinear) with infinite delay match our situation, and how they are connected to translation semigroups.

We start with the linear case. Let $F = L^1(X, \Sigma, \mu)$ where (X, Σ, μ) is a σ -finite measure space and consider as usual $E = L^1((-\infty, 0], F; e^{\eta s} ds)$. It is well known that E is isomorphic to $L^1((-\infty, 0] \times X, \mathcal{B} \otimes \Sigma, \nu \otimes \mu)$. (Here \mathcal{B} denotes the Borel algebra on $(-\infty, 0]$ and $\nu = e^{\eta s} ds$.)

Let $\{K(s): s \geq 0\}$ be a strongly continuous family of bounded linear operators on F and define $\Phi: E \rightarrow F$ by

$$(19) \quad \Phi f = \int_{-\infty}^0 K(-s)f(s) ds.$$

Assume furthermore that $\Phi \circ A_0$ has a continuous extension from E to F . We consider the renewal equation from the previous example

$$(20) \quad u(t) = \Phi(u_t), \quad u_0 = \varphi.$$

An easy manipulation of the right-hand side of (20) using the special form of Φ given in (19) leads to the following Volterra equation with values in the Banach space F and with an infinite delay:

$$(21) \quad \begin{aligned} u(t) &= \int_{-\infty}^t K(t-s)u(s) ds \quad \text{for } t > 0, \\ u(t) &= \varphi(t) \quad \text{for } t \leq 0. \end{aligned}$$

Let us now make the following assumptions concerning the operator family $\{K(s): s \geq 0\}$. For almost all $s \geq 0$ let $K(s)$ be positive and linear and $|K(s)f| \leq K|f|$ for $f \in E$, where K is a weakly compact linear operator on F . Furthermore we assume that the operators $K(s)$ for $s \geq 0$ are irreducible (see Definition 3.6).

One easily verifies that the operator defined by formula (19) is linear, positive and bounded. Moreover the operator Φ inherits the following properties from the operator family $\{K(s): s \geq 0\}$.

Lemma 7. *The operator Φ as defined in (19) is weakly compact and strictly positive. Furthermore the operators $\Phi_\lambda \in \mathcal{L}(F)$ defined by $\Phi_\lambda x = \Phi(\varepsilon_\lambda \otimes x)$ for $x \in F$ (see beginning of §3) are irreducible.*

Proof. Since $\Phi f = \int_{-\infty}^0 K(-s)f(s) ds$ we have

$$|\Phi f| \leq \int_{-\infty}^0 K(-s)|f(s)| ds \leq \int_{-\infty}^0 K|f(s)| ds = K \int_{-\infty}^0 |f(s)| ds.$$

Since K is weakly compact the right-hand side of this inequality defines a weakly compact operator from E to F . Hence we conclude that Φ is majorized by a weakly compact operator and thus is weakly compact as well. (See [1] for this and other results concerning operators dominated by weakly compact operators.)

The strict positivity of Φ is a consequence of the irreducibility of the operators $K(s)$. Let $f \in E_+$, $f \neq 0$. Then there exists a set M with positive measure such that $f(s) > 0$ whenever $s \in M$. Thus $\Phi f = \int_{-\infty}^0 K(-s)f(s) ds > 0$ by the irreducibility of $K(-s)$ for $s \in M$.

Recalling the definition of Φ_λ where $\lambda > 0$ we have for our particular Φ that

$$\Phi_\lambda x = \int_{-\infty}^0 e^{\lambda s} K(-s)x ds \quad \text{for } x \in F, \lambda > 0.$$

If $K(s)$ is irreducible for all $s \geq 0$ we have for $x \in F_+$, $x \neq 0$ and $x' \in F'_+$, $x' \neq 0$ that

$$\langle \Phi_\lambda^n x, x' \rangle = \int_{-\infty}^0 \cdots \int_{-\infty}^0 e^{\lambda s_1} \cdots e^{\lambda s_n} \langle K(-s_1) \cdots K(-s_n)x, x' \rangle ds_n \cdots ds_1 > 0$$

since $\langle K(s)^n x, x' \rangle > 0$. \square

Remark 4.5. In order to satisfy the assumptions of Proposition 3.8 we need a less restrictive assumption on the operator family $\{K(s): s \geq 0\}$. Assume that for all $s_0 \geq 0$ there exists $s \geq s_0$ such that for any $x \in F_+$, $x \neq 0$ and $x' \in F'_+$, $x' \neq 0$ we have $\langle K(s)x, x' \rangle > 0$. Then it is straightforward to show that the assumptions of Proposition 3.8 are satisfied and thus the semigroup $(T_\Phi(t))_{t \geq 0}$ is irreducible.

If we combine the result of this lemma with Theorem 4.1 we obtain

Proposition 4.6. *Let the same assumptions hold as in the preceding discussion (including the assumption either of the lemma or the remark). Suppose furthermore that $s(A_\Phi) > -\eta$. Then there exists a strictly positive unique real number $\lambda_0 > -\eta$ satisfying $r(\Phi_{\lambda_0}) = 1$ such that $\lambda_0 = s(A_\Phi)$. Moreover there exists a strictly positive projection P of rank 1 such that $\|e^{-\lambda_0 t} T_\Phi(t) - P\| \leq Me^{-\delta t}$ for suitable constants $\delta > 0$, $M \geq 0$ and all $t \geq 0$.*

EXAMPLE. A simple example for the above described type of Volterra equation is given if $K(s) = k(s)K$, where K is a weakly compact operator and $s \mapsto k(s)$ is a uniformly continuous, bounded map from \mathbb{R}_+ to \mathbb{R}_+ with unbounded support.

Let us now consider nonlinear operators Φ of the form given in formula (19) which analogously as in the linear case leads to a Volterra equation with infinite delay.

Again let $E = L^1((-\infty, 0], F; e^{\eta s} ds)$ and let $\eta > 0$. A possible choice for the operator Φ is

$$(22) \quad \Phi(f) = \int_{-\infty}^0 K(-s)g(s, f(s)) ds, \quad f \in E,$$

where g is a continuous function from $(-\infty, 0] \times F$ into F which is Fréchet-differentiable in the second variable and has a uniformly bounded derivative $D_2g(s, \bar{x})$. The assumptions on g assure the Lipschitz-continuity of Φ , thus the existence of a nonlinear semigroup $(T_\Phi(t))_{t \geq 0}$ (see Theorem 2.3).

If we want to consider the linearization of Φ in $\bar{f} = \mathbb{1} \otimes \bar{x}$ for \bar{x} satisfying $\bar{x} = \int_{-\infty}^0 K(-s)g(s, \bar{x}) ds$ we obtain that $\Phi'(\bar{f})(f) = \int_{-\infty}^0 \tilde{K}(-s)f(s) ds$ where $\tilde{K}(-s) = K(-s)D_2g(s, \bar{x})$. Here D_2g denotes the Fréchet-derivative of g in the second variable. By Proposition 3.14 and Proposition 3.15 the investigation of this linear operator $\Psi := \Phi'(\bar{f})$ from E to F and of the corresponding semigroup $(T_\Psi(t))_{t \geq 0}$ leads to results on the asymptotic properties of the equilibrium solution \bar{f} in the sense of Theorem 4.4.

As a last special case let us consider functions g such that $g(s, x)$ does not depend on s . Thus we consider functions $g(s, x) = h(x)$ where h is a continuously Fréchet-differentiable function.

Furthermore let $K(s)$ be an operator family satisfying the assumptions of Proposition 4.6 and assume that $s \mapsto K(s)$ is integrable and that

$$(23) \quad r \left(\int_{-\infty}^0 K(-s) ds \right) = 1.$$

Again let \bar{f} be an equilibrium of $(T_\Phi(t))_{t \geq 0}$, thus $\bar{f} = \mathbb{1} \otimes \bar{x}$, where

$$\bar{x} = \int_{-\infty}^0 K(-s)h(\bar{x}) ds.$$

Let $\Psi = \Phi'(\bar{f})$ be the Fréchet-derivative of Φ in \bar{f} . Thus Ψ is given by

$$\Psi(f) = h'(\bar{x}) \cdot \int_{-\infty}^0 K(-s)f(s) ds, \quad f \in E.$$

Thus Ψ is a positive operator whenever $h'(\bar{x}) \geq 0$. If $h'(\bar{x}) < 0$, then the operator $-\Psi$ is positive. In particular Ψ is dominated by $-\Psi = |\Psi|$.

Under these assumptions the results by O. Diekmann and S. A. Van Gils discussed in [31, Section VI.3] for a real valued equation of this form carry over to our situation. Since the "characteristic equation" from equivalence (10) becomes

$$1 = r(\Psi_\lambda) = h'(\bar{x}) \cdot r \left(\int_{-\infty}^0 e^{\lambda s} K(-s) ds \right)$$

if $h'(\bar{x}) \geq 0$, the spectral radius $r(\Psi_0) = 1$ given in formula (23) determines whether the unique real solution (if it exists) $\lambda_0 = s(A_\Psi)$ of $r(\Psi_\lambda) = 1$ is less, equal or bigger than 0. From this we can conclude (using Theorem 4.4) the following:

- (i) If $h'(\bar{x}) > 1$, then $s(A_\Psi) > 0$ and thus \bar{f} is unstable.
- (ii) If $0 \leq h'(\bar{x}) < 1$ then $s(A_\Psi) < 0$ and thus \bar{f} is asymptotically stable.
- (iii) If $-1 < h'(\bar{x}) < 0$ then $\omega(A_\Psi) \leq \omega(A_{|\Psi|}) = s(A_{|\Psi|}) < 0$, by Proposition 3.10, and thus \bar{f} is asymptotically stable.
- (iv) If $h'(\bar{x}) < -1$ then the information is not sufficient to decide whether \bar{f} is stable or not.

These properties are only a first step to the discussion of stability properties of equilibria \bar{f} . Nevertheless, the discussion in [31] shows nicely how, in more special cases, a detailed analysis can uncover much information about the bifurcation properties of steady states and periodic solutions.

Thus an investigation of concrete examples in our Banach space valued setting is rather promising and may be the topic of further investigations.

REFERENCES

1. C. D. Aliprantis and O. Burkinshaw, *On weakly compact operators on Banach lattices*, Proc. Amer. Math. Soc. **83** (1981), 573–578.
2. O. Arino and M. Kimmel, *Asymptotic analysis of a cell cycle model based on unequal division*, SIAM J. Appl. Math. **47** (1987), 128–145.
3. I. Becker and G. Greiner, *On the modulus of one-parameter semigroups*, Semigroup Forum **34** (1986), 185–201.
4. G. Di Blasio, K. Kunisch and E. Sinestrari, *Stability for abstract linear functional differential equations*, Israel J. Math. **50** (1985), 231–263.
5. G. Chen and R. Grimmer, *Semigroups and integral equations*, J. Integral Equations **2** (1980), 133–154.
6. Ph. Clément, O. Diekmann, M. Gyllenberg, H. J. A. M. Heijmans and H. R. Thieme, *Perturbation theory for dual semigroups*. I–V, Preprints (1986/1987).
7. C. Corduneanu and V. Lakshmikantham, *Equations with unbounded delay: a survey*, Nonlinear Analysis **4** (1980), 831–877.
8. R. Datko, *Linear autonomous neutral differential equations in a Banach space*, J. Differential Equations **25** (1977), 258–274.
9. W. Desch and W. Schappacher, *Linearized stability for nonlinear semigroups*, Differential Equations in Banach Spaces (Bologna 1985), Lecture Notes in Math., vol. 1223, Springer-Verlag, Berlin and New York, 1986, pp. 61–73.
10. O. Diekmann, *Volterra integral equations and semigroups of operators*, Math. Centre Report Amsterdam TW 197/80 (1980).
11. —, *Perturbed dual semigroups and delay equations*, Preprint (1986).
12. N. Dunford and J. T. Schwartz, *Linear operators*, Part I: *General theory*, Wiley, New York, 1958.
13. J. Dyson and R. Villella-Bressan, *Semigroups of translations associated with functional and functional differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **67** (1979), 171–188.

14. H. Flaschka and M. J. Leitman, *On semigroups of nonlinear operators and the solution of the functional differential equation $\dot{x}(t) = F(x_t)$* , J. Math. Anal. Appl. **49** (1975), 649–658.
15. J. A. Goldstein, *Nonlinear semigroups and nonlinear partial differential equations*, Atas do Décimo Colóquio Brasileiro de Matemática, Vol. I, Inst. Mat. Pura Appl., Rio de Janeiro, 1978, pp. 213–248.
16. —, *Semigroups of operators and applications. II*, Preprint (1985).
17. A. Grabosch, *A two phase transition probability model of the cell cycle*, Semesterbericht Funktionalanalysis Tübingen, 1985/86, pp. 71–99.
18. —, *Die Lösungshalbgruppe abstrakter retardierter Gleichungen*, Dissertation, Universität Tübingen, 1986.
19. G. Gripenberg, *Asymptotic behaviour of resolvents of abstract Volterra equations*, J. Math. Anal. Appl. **122** (1987), 427–438.
20. G. Greiner, *Perturbing the boundary conditions of a generator*, Houston J. Math **13** (1987).
21. —, *Semilinear boundary conditions for evolution equations*, Semesterbericht Funktionalanalysis Tübingen, 1987.
22. G. Greiner and R. Nagel, *Growth of cell populations via one-parameter semigroups of positive operators*, Semesterbericht Funktionalanalysis Tübingen, 1987.
23. J. K. Hale, *Theory of functional differential equations*, Springer-Verlag, New York and Berlin, 1977.
24. T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan **19** (1967), 508–520.
25. W. Kerscher, *Retardierte Cauchyprobleme: Ordnunseigenschaften und Stabilität unabhängig von der Verzögerung*, Dissertation, Universität Tübingen, 1986.
26. W. Kerscher and R. Nagel, *Asymptotic behavior of one-parameter semigroups of positive operators*, Acta Appl. Math. **2** (1984), 297–309.
27. —, *Positivity and stability for Cauchy problems with delay*, Semesterbericht Funktionalanalysis Tübingen, 1986, pp. 35–54.
28. M. Kimmel, Z. Darzynkiewicz, O. Arino and F. Traganos, *Analysis of a cell cycle model based on unequal division of metabolic constituents to daughter cells during cytokinesis*, J. Theoret. Biol. **101** (1984), 637–664.
29. A. Kufner, O. John and S. Fučík, *Function spaces*, Noordhoff, Leyden, 1977.
30. F. Martello, *Semigrupp di traslazioni nello spazio $L^1([-r, 0], X)$* , Rend. Sem. Mat. Univ. Padova **72** (1984), 307–317.
31. J. A. J. Metz and O. Diekmann (Eds.), *The dynamics of physiologically structured populations*, Lecture Notes in Biomath., vol. 68, Springer-Verlag, Berlin and New York, 1986.
32. R. Nagel (Ed.), *One-parameter semigroups of positive operators*, Lecture Notes in Math., vol. 1184, Springer-Verlag, Berlin and New York, 1986.
33. A. T. Plant, *Nonlinear semigroups of linear operators and applications in Banach spaces*, J. Math. Anal. Appl. **60** (1977), 67–74.
34. J. Prüß, *Equilibrium solutions of age-specific population dynamics of several species*, J. Math. Biol. **11** (1981), 65–84.
35. H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin and New York, 1974.
36. F. R. Sharpe and A. J. Lotka, *A problem in age distributions*, Philos. Mag. **21** (1911), 435–438.
37. O. J. Staffans, *The initial function and forcing function semigroups generated by a functional equation*, Preprint (1984).
38. —, *Semigroups generated by a convolution equation*, Proc. Conf. on Operator Semigroups and Applications (Graz, 1983), Lecture Notes in Math., vol. 1076, Springer-Verlag, Berlin and New York, 1984, pp. 209–226.
39. C. C. Travis and G. F. Webb, *Existence and stability for partial functional differential equations*, Trans. Amer. Math. Soc. **200** (1974), 395–418.

- 40. R. Vilella-Bressan, *Functional equation of delay type in L^1 -spaces*, Ann. Polon. Math. **45** (1985), 93–104.
- 41. G. F. Webb, *Volterra integral equations and nonlinear semigroups*, Nonlinear Analysis **1** (1977), 415–427.
- 42. —, *Theory of nonlinear age-dependent population dynamics*, Dekker, New York, 1985.
- 43. —, *An operator theoretic formulation of asynchronous exponential growth*, Preprint (1985).
- 44. —, *Random transitions, size control, and inheritance in cell population dynamics*, Preprint (1986).
- 45. G. F. Webb and A. Grabosch, *Asynchronous exponential growth in transition probability models of the cell cycle*, SIAM J. Math. Anal. **18** (1987).

DEPARTMENT OF MATHEMATICS, GEORGETOWN UNIVERSITY, WASHINGTON, D.C. 20057

Current address: Mathematisches Institut, Universität Tübingen, D-7400 Tübingen, Federal Republic of Germany